Transmission Distortion in Multimode Random Waveguides

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Abstract—We consider the coupled line equations for two-mode random media in which both modes travel in the same (forward) direction as a model for multimode millimeter waveguides and optical fibers, in which mode conversion at imperfections occurs primarily in the forward direction. Some exact general properties satisfied by the transfer function and the impulse response of such a system are given for an arbitrary coupling coefficient. A random stationary coupling coefficient with statistically independent successive values, and consequently a white spectrum (e.g., a white Gaussian or a Poisson noise), permits exact determination of transmission statistics; we obtain first- and second-order statistics in the time and frequency domains. No perturbation or other approximations are made in any of the above results, which are obtained directly from the coupled line equations. These results are used to study signal distortion in long guides.

By straightforward extension of this work more complicated calculations can treat more forward modes, but not backward modes or nonwhite coupling coefficient spectra. In this paper the coupling coefficient is assumed frequency independent, and under certain conditions the signal distortion decreases as the mode conversion increases. In practical cases the coupling coefficients are frequency dependent and the above behavior is modified; the present work is extended to this important case in a companion paper.

I. Introduction

CONSIDER the coupled line equations

$$I_0'(z) = -\Gamma_0 I_0(z) + jc(z) I_1(z)$$
$$I_1'(z) = jc(z) I_0(z) - \Gamma_1 I_1(z) \qquad (1)$$

subject to the initial conditions

$$I_0(0) = 1$$
$$I_1(0) = 0 \qquad (2)$$

the $'$ denoting differentiation with respect to z. Equation (1) describes a system of two coupled modes traveling in the forward $(+z)$ direction, with propagation constants Γ_0 and Γ_1 and complex wave amplitude $I_0(z)$ and $I_1(z)$ having time dependence $\exp(j2\pi ft)$. The real coupling coefficient $c(z)$ has arbitrary functional form. I_0 represents a desired signal mode and I_1 an undesired spurious mode. A unit signal is injected in the desired mode at $z=0$ (2); the output $I_0(z)$ is the complex signal transfer function (output/input) of the length z of the line. These equations have arisen in the study of various physical systems [1]–[3], such as circular electric mode transmission in metal waveguides [4], [5] and optical fibers [6], [7].

For example, long-distance millimeter waveguides [4], [5] use the circular electric TE-01 wave as a signal mode because of its low heat loss. Geometric imperfections, both intentional (i.e., bends) and unavoidable (e.g., random straightness deviation, diameter variation, ellipticity, etc., due to manufacturing tolerances), couple the TE-01 signal mode to other spurious propagating modes, with potentially serious effects on the signal-mode transfer function. Multimode optical fibers [6], [7] can exhibit similar behavior. In these cases the most important spurious modes travel in the forward direction (i.e., in the same direction as the signal mode) under normal conditions.

Since exact solutions to these equations can be obtained only in very special cases, perturbation theory has been widely used for approximate treatment of these equations [5]–[8]. Both deterministic and statistical $c(z)$ have been so studied.

The principal results of this paper are rigorous explicit solutions for the first- and second-order statistics of the transfer function $I_0(z)$ of (1) and (2), and of its Fourier transform (the corresponding impulse response), for stationary random coupling $c(z)$ for which $c(z_1)$ and $c(z_2)$ are statistically independent for $z_1 \neq z_2$,[1] e.g., a white Gaussian noise. This work is of value in studying the region of validity of various perturbation theories and in finding out what happens to signals in very long transmission lines. The present matrix methods have previously been used to treat active [9] and passive [10] transmission lines or equivalent one-dimensional random media, in which the spurious mode is a reflected wave, and to derive one of the results given here for the present problem [11].

In addition, we present some general properties of the transfer function $I_0(z)$ and of its associated impulse response that hold true for arbitrary (nonstatistical) coupling $c(z)$.

All of the present results are exact, being obtained directly from (1) and (2) without mathematical or physical approximations of any kind. In particular, the

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[1] Such a $c(z)$ is necessarily white, but not necessarily Gaussian (e.g., a Poisson noise).

present results hold where perturbation theory fails. Many extensions of these results—e.g., to more modes, systems that sum the output amplitudes or powers of all of the modes (as opposed to the present example where the output consists of a single mode), coupling coefficients of different form, and higher order statistics—are readily found by the present methods.

II. Notation [8]

Normalize the transfer functions $I_0(z)$ and $I_1(z)$ of (1) by

$$I_0(z) \equiv \exp(-\Gamma_0 z)\cdot G_0(z)$$
$$I_1(z) \equiv \exp(-\Gamma_1 z)\cdot G_1(z) \tag{3}$$

i.e., by removing the corresponding propagation factors for transmission through perfect guide (no coupling). Introduce the real and imaginary parts of the propagation constants as

$$\Gamma_0 \equiv \alpha_0 + j\beta_0 \qquad \Gamma_1 \equiv \alpha_1 + j\beta_1 \tag{4}$$

(i.e., the α's and β's are real). Define the differential propagation constant and its real and imaginary parts as

$$\Delta\Gamma \equiv \Gamma_0 - \Gamma_1 \equiv \Delta\alpha + j\Delta\beta \tag{5}$$
$$\Delta\alpha = \alpha_0 - \alpha_1 \leq 0$$
$$\Delta\beta = \beta_0 - \beta_1. \tag{6}$$

$\Delta\alpha$ is assumed negative because the signal mode will normally have less loss than the spurious mode. Then (1) becomes

$$G_0'(z) = jc(z)\exp(\Delta\Gamma z)\cdot G_1(z)$$
$$G_1'(z) = jc(z)\exp(-\Delta\Gamma z)\cdot G_0(z) \tag{7}$$

with initial conditions (2)

$$G_0(0) = 1$$
$$G_1(0) = 0. \tag{8}$$

The coupling coefficient $c(z)$ in practical systems [4]–[7] is proportional to some geometric imperfection, here called $d(z)$:

$$c(z) = C\cdot d(z). \tag{9}$$

$d(z)$ might be displacement, slope, or curvature of the guide axis, ellipticity of the guide, etc., depending on the application.

Next

$$G_0 \equiv 1 - A + j\theta. \tag{10}$$

The real quantities A and θ denote the departure of the real part of G_0 from unity, and the imaginary part of G_0, respectively. In the perturbation case, when G_0 departs only slightly from 1, $A \approx -\mathrm{Re}\ln G_0 =$ loss and $\theta \approx \mathrm{Im}\ln G_0 =$ phase. For the general case studied here these identifications do not hold.

Finally, we require the Fourier transform of G_0, defined as (assuming that it exists)

$$g_{\Delta\alpha}(\tau) \equiv \int_{-\infty}^{\infty} G_0(\Delta\alpha, \Delta\beta)\exp\left(-j2\pi\tau\left(\frac{\Delta\beta z}{2\pi}\right)\right) d\left(\frac{\Delta\beta z}{2\pi}\right)$$
$$= \frac{1}{2\pi}\int_{-\infty}^{\infty} G_0(\Delta\alpha, \Delta\beta)\exp(-j\tau\Delta\beta z)d(\Delta\beta z) \tag{11}$$

for fixed line length z. The inverse transform is

$$G_0(\Delta\alpha, \Delta\beta) = \int_{-\infty}^{\infty} g_{\Delta\alpha}(\tau)\exp(j\tau\Delta\beta z)d\tau. \tag{12}$$

In (11) we have chosen the normalized quantity

$$b \equiv \frac{\Delta\beta z}{2\pi} \tag{13}$$

as integration variable, rather than simply $\Delta\beta$, to normalize the range of the transform variable τ (see Section IV). $g_{\Delta\alpha}(\tau)$ of (11) turns out to be the normalized impulse response of the system under suitable conditions [see Section V, (32) and (33)].

The normalized signal transfer function G_0 is more convenient than I_0 for some purposes.

III. Application to Real Systems

For fixed length z and fixed (arbitrary) geometric parameter $d(z)$ (9), the normalized transfer function G_0 is a function of $\Delta\beta$, $\Delta\alpha$, and C, while the normalized impulse response g of (11) is a function of τ, $\Delta\alpha$, and C. Alternatively, for fixed z and random $d(z)$ with fixed statistics, the statistics of G_0 are functions of $\Delta\beta$, $\Delta\alpha$, and C, while the statistics of g depend on τ, $\Delta\alpha$, and C.

We study below exact deterministic and statistical[2] properties of $G_0(\Delta\beta)$ and $g(\tau)$, with $\Delta\alpha$ and C regarded as fixed parameters. Application of these rigorous mathematical results to practical guided-wave systems, such as circular waveguide [4], [5] and optical fibers [6], [7], involves physical approximations for several reasons.

1) In a specific guide $\Delta\alpha$, $\Delta\beta$, and C will be definite functions of frequency f, and so are related. $\Delta\alpha$ must be an even function of f, $\Delta\beta$ and C odd. However, for many purposes the variation of $\Delta\alpha$ and C with f may be neglected over suitably narrow bands. This approximation motivates choosing $\Delta\beta$ as an independent variable in the subsequent mathematical analysis, neglecting the variation of $\Delta\alpha$, and taking C of (9) as

$$C = C_0\cdot\mathrm{sgn}\, f$$
$$= \begin{cases} C_0, & f > 0, \\ -C_0, & f < 0, \end{cases} \quad C_0 \text{ a positive constant.} \tag{14}$$ [3]

[2] For white $d(z)$ with independent successive values.

[3] When $\Delta\beta$ is taken as the independent variable, $\Delta\beta$ may be substituted for f in (14).

2) Practical systems have many spurious modes. Moreover the spurious modes travel in both (forward and backward) directions.

Extension of the present analysis is required to remove these two types of limitations. Practical utility of the present results clearly depends on the specific application.

The phase constants β_0 and β_1 are in general nonlinear functions of frequency f; therefore, $\Delta\beta$ is also in general a nonlinear function of f. A further approximation that is often useful is to regard $\Delta\beta$ as a linear function of f over a suitably narrow band. Recalling that phase and group velocities of each mode are defined as

$$v_\varphi \equiv \frac{\omega}{\beta} \qquad v_g \equiv \frac{d\omega}{d\beta} \qquad \omega \equiv 2\pi f$$

$\Delta\beta$ will be approximately linear with f if the group velocities are essentially constant over the band. Under these conditions τ is a normalized time variable. Appendix I gives the frequency dependence of $\Delta\beta$ for two cases used as examples below.

IV. General Properties of G_0 and g

Assume that the geometric imperfection $d(z)$ is some fixed but arbitrary function and that the overall length z of guide is fixed. Under these conditions $G_0(\Delta\alpha, \Delta\beta)$, the solution to (7) and (8), and its Fourier transform $g_{\Delta\alpha}(\tau)$, defined by (11), have the following properties, without approximation.

1) G_0 is analytic in the complex $\Delta\Gamma$ plane; i.e., G_0 has no singularities for any finite $\Delta\Gamma$.

2) $g_{\Delta\alpha}(\tau)$ is causal; i.e.,

$$g_{\Delta\alpha}(\tau) = 0, \qquad \tau < 0. \tag{15}$$

This implies that A and θ are Hilbert transforms:[4]

$$\theta = -\hat{A} \tag{16}$$

where $\hat{}$ denotes the Hilbert transform [12],

$$\begin{aligned} \hat{A}(\Delta\alpha, \Delta\beta) &= \frac{1}{\pi} A(\Delta\alpha, \Delta\beta) * \frac{1}{\Delta\beta} \\ &= \frac{1}{\pi}\int_{-\infty}^{\infty} \frac{A(\Delta\alpha, \sigma)}{\Delta\beta - \sigma} d\sigma \end{aligned} \tag{17}$$

$*$ representing the convolution operator [12]. The inverse relationship is

$$A = \hat{\theta} + \overline{A}, \qquad \overline{A} \equiv \lim_{B\to\infty} \frac{1}{2B}\int_{-B}^{B} A(\Delta\alpha, \sigma) d\sigma \tag{18}$$

the constant $\overline{A}$ in (18) representing the dc component of A, not recovered by the Hilbert transform. Equations (15)–(18) are true for every (fixed) $\Delta\alpha$.

[4] Previously shown in the perturbation case [13]; the present results are general.

3) G_0 and g for any negative $\Delta\alpha$ are simply related to the corresponding quantities for $\Delta\alpha = 0$:

$$g_{\Delta\alpha}(\tau) = \exp(\Delta\alpha z\tau) g_0(\tau) = \exp(-|\Delta\alpha| z\tau) g_0(\tau), \qquad \Delta\alpha \le 0 \tag{19}$$

where $g_0(\tau) = g_{\Delta\alpha}(\tau)\big|_{\Delta\alpha=0}$. Therefore,

$$G_0(\Delta\alpha, \Delta\beta) = \frac{1}{2\pi} G_0(0, \Delta\beta) * \frac{1}{|\Delta\alpha| - j\Delta\beta}, \qquad \Delta\alpha \le 0. \tag{20}$$

Then

$$A(\Delta\alpha, \Delta\beta) = \frac{1}{\pi} A(0, \Delta\beta) * \frac{|\Delta\alpha|}{\Delta\alpha^2 + \Delta\beta^2}, \qquad \Delta\alpha \le 0. \tag{21}$$ [5]

$$\theta(\Delta\alpha, \Delta\beta) = \frac{1}{\pi} \theta(0, \Delta\beta) * \frac{|\Delta\alpha|}{\Delta\alpha^2 + \Delta\beta^2}, \qquad \Delta\alpha \le 0. \tag{22}$$ [5]

Increasing the differential loss (making $\Delta\alpha$ more negative) smooths out the transfer-function fluctuations by convolution with a simple window function. The window function of (21) and (22) has unit area

$$\frac{1}{\pi|\Delta\alpha|}\int_{-\infty}^{\infty} \frac{1}{1 + \left(\frac{\Delta\beta}{\Delta\alpha}\right)^2} d(\Delta\beta) = 1; \tag{23}$$

it becomes a unit impulse as $\Delta\alpha \to 0$

$$\lim_{\Delta\alpha\to 0} \frac{1}{\pi|\Delta\alpha|} \frac{1}{1 + \left(\frac{\Delta\beta}{\Delta\alpha}\right)^2} = \delta(\Delta\beta) \tag{24}$$

as required for consistency in (21) and (22).

4) $G_0(\Delta\alpha, \Delta\beta)$ is band-limited, i.e., determined by its values at sample points

$$\Delta\beta_n = \frac{2\pi n}{z} \tag{25}$$

by the relation

$$G_0(\Delta\alpha, \Delta\beta) = \exp\left(j\pi\frac{\Delta\beta z}{2\pi}\right) \sum_{n=-\infty}^{\infty} G_0\left(\Delta\alpha, \frac{2\pi n}{z}\right)(-1)^n \cdot \frac{\sin\pi\left(\frac{\Delta\beta z}{2\pi} - n\right)}{\pi\left(\frac{\Delta\beta z}{2\pi} - n\right)} \tag{26}$$

since the normalized impulse response is time-limited,

$$g_{\Delta\alpha}(\tau) = 0, \qquad \tau < 0, \quad \tau > 1. \tag{27}$$

[5] Previously shown in the perturbation case [13]; the present results are general.

5) $g_{\Delta\alpha}(\tau)$ is real, and

$$G_0(\Delta\alpha, -\Delta\beta) = G_0^*(\Delta\alpha, \Delta\beta)$$
$$A(\Delta\alpha, -\Delta\beta) = A(\Delta\alpha, \Delta\beta)$$
$$\theta(\Delta\alpha, -\Delta\beta) = -\theta(\Delta\alpha, \Delta\beta)$$
$$\bar{\theta} \equiv \lim_{B\to\infty} \frac{1}{2B}\int_{-B}^{B} \theta(\Delta\alpha, \sigma)d\sigma = 0. \tag{28}$$

Introduction of the normalized variables [see (13)]

$$a \equiv \frac{\Delta\alpha z}{2\pi} \qquad b \equiv \frac{\Delta\beta z}{2\pi} \tag{29}$$

leaves the above results involving convolution (including Hilbert transforms) unchanged in form.

These results are derived in Appendix II. Their physical interpretation is discussed in Section V. They are reflected in the statistical results derived in Sections VIII–X.

V. Discussion of Section IV

We give a simple physical interpretation for the mathematical properties of Section IV in the case discussed in Section III, dispersionless modes and frequency-independent attenuation constants and coupling coefficient; i.e., we assume $\Delta\beta \propto f$ (149), and $\Delta\alpha$ (6) and C_0 (14) independent of f, over the infinite frequency range $-\infty < f < \infty$. Of course, we have not demonstrated a circuit model governed by (1) and these assumptions.

Let the transfer function as a function of frequency f be denoted by $\mathcal{G}_{\Delta\alpha}(f)$, related to G_0 of (3) by

$$\mathcal{G}_{\Delta\alpha}(f) \equiv G_0(\Delta\alpha, \Delta\beta). \tag{30}$$

The impulse response $\wp_{\Delta\alpha}(t)$ in the time domain and $\mathcal{G}_{\Delta\alpha}(f)$ are Fourier transforms:

$$\wp_{\Delta\alpha}(t) = \int_{-\infty}^{\infty} \mathcal{G}_{\Delta\alpha}(f) \exp(j2\pi ft)df. \tag{31}$$

Then using (149) the impulse response of (31) is related to the normalized impulse response of (11) by

$$\wp_{\Delta\alpha}(t) = \frac{1}{T} g_{\Delta\alpha}\left(\frac{t}{T}\right) \tag{32}$$

where

$$T \equiv z\left(\frac{1}{v_1} - \frac{1}{v_0}\right) \tag{33}$$

is the delay between signal and spurious modes for a length z of transmission line. By (32) and (27) the impulse response $\wp_{\Delta\alpha}(t)$ is limited in duration to T s.

The coupling coefficient for an elementary length of line is, from (9) and (14), $jc(x)dx = j \operatorname{sgn} f \cdot C_0 d(x)dx$. A unit impulse $\delta(t)$ traveling in one mode, incident on this elementary mode converter, excites an amplitude

$$-C_0 d(x)dx \cdot \frac{1}{\pi t} \tag{34a}$$

in the other mode;[6] similarly the wave $-(1/\pi t)$ traveling in one mode, incident on this elementary converter, excites an amplitude

$$-C_0 d(x)dx \cdot \delta(t) \tag{34b}$$

in the other mode. Consider a unit impulse in the signal mode incident on the coupled line described by (1) at $z=0$. The integrand of (154), with $c(z)$ replaced by (9) and (14), corresponds to the partial impulse response (in the signal mode for a line length z)

$$\delta\left(t - \frac{x_1 - x_2 + \cdots - x_{2n}}{z} T\right) \cdot \exp[\Delta\alpha(x_1 - x_2 + \cdots - x_{2n})] \cdot C_0^{2n} d(x_1)d(x_2) \cdots d(x_{2n}) \cdot dx_1 dx_2 \cdots dx_{2n}, \quad x_1 > x_2 > \cdots > x_{2n} \tag{35}$$

i.e., that portion of the input impulse that has made exactly $2n$ transitions, from signal to spurious mode at $x_{2n}, \cdots, x_4, x_2$ and from spurious to signal mode at $x_{2n-1}, \cdots, x_3, x_1$. Summing up all such contributions, the total signal–signal impulse response is clearly causal, time-limited, and real.[7] Moreover (35) shows that for $\Delta\alpha < 0$ the damping (exponential) factor is the same for the response at a given time, whatever the number of transitions $2n$; stated differently, all portions of the total impulse response arriving at time t have traveled in the spurious mode for a total distance zt/T, independently of n. Therefore, if the geometric imperfection $d(z)$ and the coupling $c(z)$ are held fixed while the differential attenuation $\Delta\alpha$ is varied, the impulse response as a function of $\Delta\alpha$ is given as

$$\wp_{\Delta\alpha}(t) = \exp(-|\Delta\alpha| zt/T) \cdot \wp_0(t), \qquad \Delta\alpha \le 0 \tag{36}$$

corresponding to (19).

Equations (32), (33), (35), and (36), for the signal–signal response, are true only if the assumptions of (147) are satisfied and if C_0 and $\Delta\alpha$ are strictly independent of frequency.

VI. Matrix Formulation

We now turn to the treatment of the coupled line equations (1) for a stationary random coupling coefficient $c(z)$ and geometric imperfection $d(z)$ with in-

[6] The inverse Fourier transform of $-j \operatorname{sgn} f$ is $1/\pi t$, the Hilbert transform of $\delta(t)$ [12].

[7] In contrast, the signal-spurious mode response is obviously not causal, from (34a). Consequently, no circuit model can be devised that yields (1). This nonphysical behavior arises because the coupling coefficient C_0 of (14) has been assumed independent of frequency. Nevertheless, a frequency-independent coupling coefficient is often assumed, and the present results are useful approximations over limited bands. Treatment of frequency-dependent C_0 is given in a companion paper [20] as a simple extension of this work.

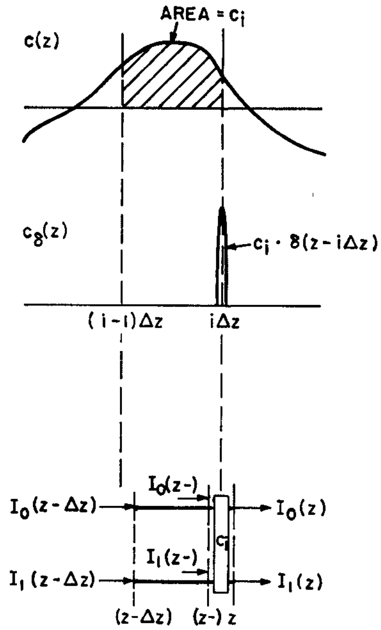

Fig. 1. Continuous line as limit of discrete sections.

dependent successive values, e.g., white Gaussian noise. We first approximate the line by a discrete model, obtained by dividing the line up into sections of length Δz and approximating $c(z)$ in each section by a function that permits exact solution of (1) within each section. Suitable approximations are step or δ functions because (1) may be solved for $c(z) = \text{constant}$ [1] or $c(z) = \text{constant} \cdot \delta(z)$ [5]. While step-function approximation for $c(z)$ has been used for a similar purpose [11], δ-function approximation is used here because it is simpler [10]. The solution for the entire line may now be expressed in terms of the known exact solutions for the individual sections by matrix techniques. The different sections are statistically independent because $c(z_1)$ and $c(z_2)$ are assumed independent for all $z_1 \neq z_2$. Exact transmission statistics for the discrete model can consequently be computed by using Kronecker products [16]. Finally, we allow $\Delta z \to 0$, thus obtaining exact transmission statistics for (1).

Thus approximate the coupling $c(z)$ in (1) by

$$c_\delta(z) = \sum_{i=1}^{(z/\Delta z)} c_i \cdot \delta(z - i\Delta z) \tag{37}$$

where δ represents the unit impulse and

$$c_i \equiv \int_{(i-1)\Delta z}^{i\Delta z} c(z)\, dz = C \cdot d_i$$

$$d_i \equiv \int_{(i-1)\Delta z}^{i\Delta z} d(z)\, dz. \tag{38}$$

The approximation becomes exact in the limit, i.e., $c_\delta(z) \to c(z)$ as $\Delta z \to 0$. The ith section of the discrete model consists of an ideal line section Δz long with zero coupling, terminated in a discrete (δ-function) mode converter of magnitude c_i, as shown in Fig. 1. The ideal line section is governed by (1) with $c(z) = 0$:

$$\begin{bmatrix} I_0(z-) \\ I_1(z-) \end{bmatrix} = \begin{bmatrix} \exp(-\Gamma_0 \Delta z) & 0 \\ 0 & \exp(-\Gamma_1 \Delta z) \end{bmatrix} \begin{bmatrix} I_0(z - \Delta z) \\ I_1(z - \Delta z) \end{bmatrix}. \tag{39}$$

The discrete converter is governed by the solution of (1) with $c(z)$ a δ function. This has been obtained from the solution to these equations for constant $c(z)$ [5], i.e., by setting $c(z) = c_i/\Delta$ for a range of z of length Δ and allowing $\Delta \to 0$, yielding

$$\begin{bmatrix} I_0(z) \\ I_1(z) \end{bmatrix} = \begin{bmatrix} \cos c_i & j \sin c_i \\ j \sin c_i & \cos c_i \end{bmatrix} \begin{bmatrix} I_0(z-) \\ I_1(z-) \end{bmatrix}. \tag{40}$$

Combining these relations and introducing matrix notation,

$$\mathfrak{I}(z) = \mathfrak{X}(z)\mathfrak{I}(z - \Delta z) \tag{41}$$

where

$$\mathfrak{I}(z) \equiv \begin{bmatrix} I_0(z) \\ I_1(z) \end{bmatrix} \tag{42}$$

$$\mathfrak{X}(z) \equiv \begin{bmatrix} \exp(-\Gamma_0 \Delta z) \cos c_i & \exp(-\Gamma_1 \Delta z) j \sin c_i \\ \exp(-\Gamma_0 \Delta z) j \sin c_i & \exp(-\Gamma_1 \Delta z) \cos c_i \end{bmatrix} \tag{43}$$

and c_i is given by (38). The initial conditions of (2) become

$$\mathfrak{I}(0) = \begin{bmatrix} 1 \\ 0 \end{bmatrix}. \tag{44}$$

The solution to (41)–(44) and (38) should provide a good approximation to the solution of (1) and (2) when Δz becomes small compared to the beat wavelength, i.e.,

$$\Delta z \ll \frac{1}{|\Delta\Gamma|} = \frac{1}{|\Gamma_0 - \Gamma_1|} \tag{45}$$

providing an exact solution as $\Delta z \to 0$.

It proves convenient for some of the following calculations to normalize the above relations in a different way than in (3). Define the column vector

$$\mathfrak{G}(z) \equiv \begin{bmatrix} G_0(z) \\ \exp(\Delta\Gamma z) G_1(z) \end{bmatrix} = \begin{bmatrix} G_0(z) \\ \mathbf{G}_1(z) \end{bmatrix}$$

$$\mathbf{G}_1(z) \equiv \exp(\Delta\Gamma z) G_1(z) = \exp(\Gamma_0 z) I_1(z) \tag{46}$$

G_0 and G_1 being defined in (3). G_0 is the normalized signal transfer function defined before, while the new quantity $\mathbf{G}_1$ is different from G_1 of (3). It seems natural to refer to $\mathbf{G}_1$ as the normalized signal-spurious mode transfer function. Define the matrix

$$\mathfrak{Y}(z) \equiv \begin{bmatrix} \cos c_i & \exp(\Delta\Gamma \cdot \Delta z) j \sin c_i \\ j \sin c_i & \exp(\Delta\Gamma \cdot \Delta z) \cos c_i \end{bmatrix} \tag{47}$$

and recall that c_i is given by (38). Then

$$\mathfrak{I}(z) = \exp(-\Gamma_0 z)\mathfrak{G}(z) \qquad \mathfrak{X}(z) = \exp(-\Gamma_0 \Delta z)\mathfrak{Y}(z) \tag{48}$$

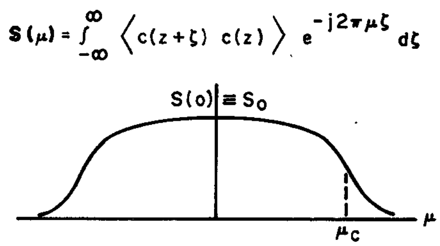

Fig. 2. Spectral density of coupling coefficient.

and (41) and (44) become

$$\mathfrak{G}(z) = \mathfrak{Y}(z)\mathfrak{G}(z - \Delta z) \qquad \mathfrak{G}(0) = \begin{bmatrix} 1 \\ 0 \end{bmatrix}. \tag{49}$$

Equation (49) is used below to obtain directly statistics of the normalized (signal) transfer function G_0 (3).

VII. Statistical Model

Assume that the random geometric imperfection $d(z)$ of (9) is stationary with symmetric probability density, and possesses the property that $d(z_1)$ and $d(z_2)$ are statistically independent for $z_1 \neq z_2$. Consequently, $d(z)$ has a white spectrum (with infinite bandwidth), with spectral density denoted by D_0:

$$\langle d(z + \zeta)d(z)\rangle = D_0 \cdot \delta(\zeta). \tag{50}$$

δ represents the unit impulse. Assuming frequency-independent coupling (14), the coupling $c(z)$ (9) has covariance

$$\langle c(z + \zeta)c(z)\rangle = S_0 \cdot \delta(\zeta), \qquad S_0 = C_0^2 D_0 \tag{51}$$

and thus also a white spectrum, with spectral density S_0. Therefore, the c_i or d_i of Section VI are independent symmetrical random variables with zero mean and variance determined from (38) and (51):

$$\begin{aligned} \langle c_i^2\rangle &= S_0 \cdot \Delta z & \langle d_i^2\rangle &= D_0 \cdot \Delta z \\ \langle c_i\rangle &= 0 & \langle d_i\rangle &= 0 \\ \langle c_i c_j\rangle &= 0 & \langle d_i d_j\rangle &= 0, \qquad i \neq j. \end{aligned} \tag{52}$$

A particular example of interest is the case of white Gaussian $d(z)$ and $c(z)$ with zero mean; then the c_i and d_i of (52) are independent Gaussian random variables with zero mean. The statistics of the random matrices $\mathfrak{X}(z)$ and $\mathfrak{Y}(z)$ [(43) and (47)] are readily determined from (52); $\mathfrak{X}$'s (or $\mathfrak{Y}$'s) corresponding to different line sections $(i \neq j)$ are statistically independent.

The assumed white spectra for $c(z)$ and $d(z)$ correspond to a coupling and geometric imperfection with infinite power. We show that the present analysis for strictly white imperfections applies also to imperfections with low-pass spectra under suitable conditions.

Fig. 2 shows a typical low-pass spectrum for $c(z)$, with low-frequency asymptote S_0 and cutoff spatial frequency μ_c. We need to divide the line into sections Δz short compared to the beat wavelength in order that the discrete approximation of Section VI will be valid, but long enough so that different sections are statistically independent in order that the subsequent statistical analysis can be readily performed. The first requirement leads to (45) and the second to the restriction $\Delta z \gg 1/\mu_c$. Combining these relationships,

$$\mu_c \gg |\Delta\Gamma|. \tag{53}$$

The following results for a strictly white coupling $(\mu_c \rightarrow \infty)$ will provide a good approximation for a low-pass coupling spectrum (finite μ_c in Fig. 2) if (53) is satisfied.

VIII. Expected Response

We first determine the expected values $\langle I_0(z)\rangle$ and $\langle I_1(z)\rangle$ of signal- and spurious-mode (complex) transfer functions of (1) with initial conditions of (2) for white $c(z)$ (50) and (51) with independent successive values. Take the expected value of both sides of (41) [16], using (42) and (43), and noting that $\mathfrak{X}(z)$ is independent of $\mathfrak{J}(z-\Delta z)$ by the independence of the c_i,

$$\begin{aligned} \langle I_0(z)\rangle &= \exp(-\Gamma_0\Delta z)\langle\cos c_i\rangle\langle I_0(z - \Delta z)\rangle \\ &= [1 - \Gamma_0\Delta z - \tfrac{1}{2}\langle c_i^2\rangle + \cdots]\langle I_0(z - \Delta z)\rangle \\ \langle I_1(z)\rangle &= [1 - \Gamma_1\Delta z - \tfrac{1}{2}\langle c_i^2\rangle + \cdots]\langle I_1(z - \Delta z)\rangle. \end{aligned} \tag{54}$$ [8]

Using (52) and taking the limit as $\Delta z \rightarrow 0$,

$$\begin{aligned} \langle I_0(z)\rangle' &= -\left(\Gamma_0 + \frac{S_0}{2}\right)\langle I_0(z)\rangle \\ \langle I_1(z)\rangle' &= -\left(\Gamma_1 + \frac{S_0}{2}\right)\langle I_1(z)\rangle \end{aligned} \tag{55}$$

the $'$ again denoting differentiation with respect to z. Using the initial conditions (2) or (44) the solution of (55) is

$$\begin{aligned} \langle I_0(z)\rangle &= \exp(-\Gamma_0 z)\exp\left(-\frac{S_0}{2}z\right) & \langle I_1(z)\rangle &= 0 \\ \langle G_0(z)\rangle &= \exp\left(-\frac{S_0}{2}z\right) & \langle G_1(z)\rangle &= 0 \end{aligned} \tag{56}$$

where we have used (3).

The expected complex signal and spurious waves have simple exponential behavior, the exponents being the respective propagation constants in perfect line [i.e., $c(z) = 0$] plus $S_0/2$, S_0 being the spectral density of $c(z)$.[9] The larger the coupling, the sooner the expected values decay to zero. The present initial conditions (zero spurious-mode input) cause the expected spurious wave to be identically zero for all z. The results of (56) are

[8] The higher terms, indicated by $\cdots$, are of higher order in Δz for white Gaussian $c(z)$, and so disappear as $\Delta z \rightarrow 0$. The same results apply to Poisson $c(z)$ only if the individual δ functions that comprise $c(z)$ have areas small compared to (1); otherwise, minor modifications must be made.

[9] This behavior differs from that for one-dimensional random media, such as layered media or random TEM transmission lines, the spurious mode being a reflected wave, where the expected complex waves vary precisely as do corresponding waves in perfect line [10].

consistent with perturbation theory [5] for $S_0 z \ll 1$, but are of course exact for all $S_0 z$.

These expected values, while of interest, give only limited information about the transmission statistics. For example, it is reasonable to suppose that for large z the spurious mode $I_1(z)$ has uniformly distributed phase. This is consistent with zero expected value (56); however, the mean square value or average spurious-mode power $\langle |I_1|^2 \rangle$ will of course not be zero. We clearly require higher order statistics of I_0 and I_1. These are readily computed via Kronecker products [16]. Exact second-order transmission statistics are found in Section IX.

IX. Second-Order Transfer-Function Statistics

We now seek the second-order statistics of the signal transfer function, and choose for convenience to work with the normalized G_0 rather than I_0. Thus we seek the covariance

$$R_0(\sigma) \equiv \langle G_0(\Delta\beta + \sigma) G_0^*(\Delta\beta) \rangle \tag{57}$$

for white $c(z)$ with independent successive values. We define the following auxiliary quantities that appear in the analysis, although they are not of direct interest for our present purposes:

$$\begin{aligned} R_{01}(\sigma) &\equiv \langle G_0(\Delta\beta + \sigma) G_1^*(\Delta\beta) \rangle \\ &= \exp(\Delta\alpha z) \exp(-j\Delta\beta z) \langle G_0(\Delta\beta + \sigma) G_1^*(\Delta\beta) \rangle \end{aligned} \tag{58}$$

$$\begin{aligned} R_{10}(\sigma) &\equiv \langle G_1(\Delta\beta + \sigma) G_0^*(\Delta\beta) \rangle \\ &= \exp(\Delta\alpha z) \exp(j\Delta\beta z) \exp(j\sigma z) \\ &\quad \cdot \langle G_1(\Delta\beta + \sigma) G_0^*(\Delta\beta) \rangle \end{aligned} \tag{59}$$

$$\begin{aligned} R_1(\sigma) &\equiv \langle G_1(\Delta\beta + \sigma) G_1^*(\Delta\beta) \rangle \\ &= \exp(2\Delta\alpha z) \exp(j\sigma z) \langle G_1(\Delta\beta + \sigma) G_1^*(\Delta\beta) \rangle. \end{aligned} \tag{60}$$

We regard $\Delta\alpha$ (6) and C_0 (14) as fixed parameters (Section III). We define the column vector $\mathfrak{R}$ as

$$\mathfrak{R}(\sigma) \equiv \begin{bmatrix} R_0(\sigma) \\ R_{01}(\sigma) \\ R_{10}(\sigma) \\ R_1(\sigma) \end{bmatrix}. \tag{61}$$

Define the column vector $\mathfrak{G}_\sigma$, the matrix $\mathfrak{Y}_\sigma$, and C_σ as given by (46), (47), and (14), respectively, with $\Delta\beta \rightarrow \Delta\beta + \sigma$ throughout.[10] Indicate the z dependence of $R(\sigma)$ of (61) by appending the subscript z. Then from (46)

$$\mathfrak{R}_z(\sigma) = \langle \mathfrak{G}_\sigma(z) \times \mathfrak{G}^*(z) \rangle \tag{62}$$

the symbol $\times$ indicating the Kronecker product [16]. The initial conditions of (49) become

$$\mathfrak{R}_0(\sigma) = \begin{bmatrix} 1 \\ 0 \\ 0 \\ 0 \end{bmatrix}. \tag{63}$$

From (49)

$$\begin{aligned} &\mathfrak{G}_\sigma(z) \times \mathfrak{G}^*(z) \\ &= (\mathfrak{Y}_\sigma(z)\mathfrak{G}_\sigma(z - \Delta z)) \times (\mathfrak{Y}^*(z)\mathfrak{G}^*(z - \Delta z)) \\ &= (\mathfrak{Y}_\sigma(z) \times \mathfrak{Y}^*(z))(\mathfrak{G}_\sigma(z - \Delta z) \times \mathfrak{G}^*(z - \Delta z)) \end{aligned} \tag{64}$$

where we use the fact that the Kronecker product of (ordinary) matrix products equals the matrix product of Kronecker products [16]. Taking expected values on both sides of (64), noting that the first and second factors of the final expression in (64) are independent by the independence of the c_i, and using (62),

$$\mathfrak{R}_z(\sigma) = \langle \mathfrak{Y}_\sigma(z) \times \mathfrak{Y}^*(z) \rangle \mathfrak{R}_{z-\Delta z}(\sigma). \tag{65}$$

We write out the first factor from (47) as

$$\langle \mathfrak{Y}_\sigma(z) \times \mathfrak{Y}^*(z) \rangle$$
$$= \begin{bmatrix} \langle \cos^2 c_i \rangle & 0 & 0 & \pm \exp[(2\Delta\alpha + j\sigma)\Delta z] \langle \sin^2 c_i \rangle \\ 0 & \exp(\Delta\Gamma^* \Delta z) \langle \cos^2 c_i \rangle & \pm \exp[(\Delta\Gamma + j\sigma)\Delta z] \langle \sin^2 c_i \rangle & 0 \\ 0 & \pm \exp(\Delta\Gamma^* \Delta z) \langle \sin^2 c_i \rangle & \exp[(\Delta\Gamma + j\sigma)\Delta z] \langle \cos^2 c_i \rangle & 0 \\ \pm \langle \sin^2 c_i \rangle & 0 & 0 & \exp[(2\Delta\alpha + j\sigma)\Delta z] \langle \cos^2 c_i \rangle \end{bmatrix} \tag{66}$$

where we have replaced C_σ by $\pm C$ according to the previous footnote; the zero terms arise from the symmetry of the c_i. Using (52), we set

$$\begin{aligned} \langle \cos^2 c_i \rangle &= 1 - \langle c_i^2 \rangle + \cdots = 1 - S_0 \Delta z + \cdots \\ \langle \sin^2 c_i \rangle &= \langle c_i^2 \rangle + \cdots = S_0 \Delta z + \cdots \end{aligned} \tag{67}$$ [11]

throughout (66), and take the limit of (65) as $\Delta z \rightarrow 0$. Equation (65) splits up into the following two uncoupled pairs of equations, in which $'$ denotes differentiation with respect to z; the initial conditions for each pair are obtained from (63):

$$\begin{aligned} R_0' &= -S_0 R_0 \pm S_0 R_1 \\ R_1' &= \pm S_0 R_0 + (2\Delta\alpha + j\sigma - S_0) R_1 \\ R_0(\sigma)\big|_{z=0} &= 1 \qquad R_1(\sigma)\big|_{z=0} = 0. \end{aligned} \tag{68}$$ [12]

[10] Note that $C_\sigma = C$ if $\Delta\beta$ and $\Delta\beta + \sigma$ have the same sign, $C_\sigma = -C$ if $\Delta\beta$ and $\Delta\beta + \sigma$ have the opposite sign.

[11] See footnote to (54).

[12] The upper (+) signs apply in the usual case where $\Delta\beta$ and $\Delta\beta + \sigma$ have the same sign (e.g., the two frequencies at which the covariance is computed are both positive); the lower (−) signs apply when $\Delta\beta$ and $\Delta\beta + \sigma$ have the opposite sign.

$$R_{01}' = (\Delta\Gamma^* - S_0)R_{01} \pm S_0 R_{10}$$
$$R_{10}' = \pm S_0 R_{01} + (\Delta\Gamma + j\sigma - S_0)R_{10}$$
$$R_{01}(\sigma)\big|_{z=0} = 0 \qquad R_{10}(\sigma)\big|_{z=0} = 0. \tag{69}$$
[12]

Equation (69) has the trivial solution

$$R_{01}(\sigma) = 0 \qquad R_{10}(\sigma) = 0$$

for all z. Consequently, signal and spurious waves remain uncorrelated because of our initial assumption (2), used throughout, of unit-input signal and zero-input spurious-wave amplitudes. Thus only (68) is pertinent here. Equation (69) may become pertinent with different initial conditions in related problems where there are inputs in both modes and where the desired response is the sum of amplitudes of the two modes.

The solution to (68) yields exact expressions for $R_0(\sigma)$ (57), the covariance of the normalized signal transfer function, and $R_1(\sigma)$ (60), the covariance of the normalized signal-spurious mode transfer function, for white coupling (50) and (51):

$$R_0(\sigma) = \exp\left[-S_0 z(1-\Sigma)\right]\left[\cosh\left(S_0 z\sqrt{1+\Sigma^2}\right) - \Sigma\,\frac{\sinh\left(S_0 z\sqrt{1+\Sigma^2}\right)}{\sqrt{1+\Sigma^2}}\right]$$

$$R_1(\sigma) = \pm\exp\left[-S_0 z(1-\Sigma)\right]\frac{\sinh\left(S_0 z\sqrt{1+\Sigma^2}\right)}{\sqrt{1+\Sigma^2}}$$

$$\Sigma \equiv \frac{\Delta\alpha + j\sigma/2}{S_0} \tag{70}$$
[12]

where we have defined the new normalized quantity Σ for economy of notation. Perturbation results [5] are obtained from these general (exact) relations by setting $S_0 z \ll 1$ and making suitable approximations.

Equation (70)—the main result of this paper—gives the second-order statistics of the two-mode transmission system of (1) for white coupling $c(z)$ with independent successive values, e.g., white Gaussian $c(z)$. We discuss below only a few representative results that can be obtained from these relations; in particular, we restrict our attention to the signal–signal transfer function I_0 or G_0, and so are interested only in $R_0(\sigma)$ for the remainder of this paper.

$R_0(\sigma)$ yields the signal (second-order) transmission statistics in the frequency (transfer-function) domain. An equivalent description in the time domain is given in Section X. A number of examples appear in Section XI.

The case $\sigma = 0$ yields the average powers in the two modes, and so is of special interest. Denoting the powers in the two modes by

$$P_0 \equiv \langle |I_0|^2\rangle \qquad P_1 \equiv \langle |I_1|^2\rangle \tag{71}$$

we have, setting $\sigma = 0$ in (57) and (60) and using (3),

$$P_0 = \exp(-2\alpha_0 z)\cdot R_0(0) \quad P_1 = \exp(-2\alpha_0 z)\cdot R_1(0) \tag{72}$$

the arguments of the R's representing $\sigma = 0$. Substituting into (68) with $\sigma = 0$,[13] we find

$$P_0' = -(2\alpha_0 + S_0)P_0 + S_0 P_1$$
$$P_1' = S_0 P_0 - (2\alpha_1 + S_0)P_1 \tag{73}$$

as previously obtained [11].[14] Adding the two relations of (73),

$$(P_0 + P_1)' = -2\alpha_0 P_0 - 2\alpha_1 P_1. \tag{74}$$

A physical interpretation of (73) says that the power in each mode decreases due to heat loss and conversion to the other mode, and increases due to conversion from the other mode. Equation (74) shows that each mode contributes to the decrease in total power flowing along the guide in proportion to its attenuation constant; this result has previously been obtained directly from (1) [5]. The solutions to (73) are readily obtained from (70) with $\sigma = 0$ or directly; they are consistent with results of perturbation theory [5] for $S_0 \ll 1$, but are of course exact for all $S_0 z$.

X. Frequency- and Time-Domain Response Statistics

Consider a filter with transfer function $\mathcal{G}(f)$ and impulse response $\mathcal{g}(t)$:

$$\mathcal{g}(t) = \int_{-\infty}^{\infty} \mathcal{G}(f)\exp(j2\pi ft)\,df. \tag{75}$$

We use script notation ($\mathcal{g}$ and $\mathcal{G}$), as in Section V, to indicate functions of actual (i.e., *not* normalized) time and frequency. Assume that the system is real and causal:

$$\mathcal{g}(t) = \mathcal{g}^*(t) \tag{76}$$
$$\mathcal{g}(t) = 0, \qquad t < 0. \tag{77}$$

Therefore,

$$\mathcal{G}(f) = \mathcal{G}^*(-f) \tag{78}$$
$$\mathcal{G}(f) = \int_0^{\infty} g(t)\exp(-j2\pi ft)\,dt. \tag{79}$$

Define [compare (10)]

$$\mathcal{G}(f) \equiv 1 - \mathcal{a}(f) + j\Theta(f) \tag{80}$$

$\mathcal{a}$ and Θ being real. From (78)

$$\mathcal{a}(f) = \mathcal{a}(-f) \qquad \Theta(f) = -\Theta(-f). \tag{81}$$

Let the symbol $\overline{}$ denote an average over frequency. Thus the dc component of $\mathcal{a}$ is

$$\bar{\mathcal{a}} \equiv \lim_{F\to\infty}\frac{1}{2F}\int_{-F}^{F}\mathcal{a}(f)\,df. \tag{82}$$

[13] Note that the upper (+) signs always apply when $\sigma = 0$.

[14] Alternatively (73) may be obtained directly from (41) by taking the Kronecker product of this relation with its conjugate.

It is convenient to define the ac component of $\mathcal{A}$ by

$$\mathcal{A}(f) \equiv \bar{\mathcal{A}} + \mathcal{A}_{ac}(f). \qquad (83)$$

From the second relation of (81),

$$\bar{\Theta} = 0. \qquad (84)$$

Since the dc component of Θ is zero, distinctive notation for its ac component is unnecessary. From (77) or (79) we have the Hilbert transform relationships [12]

$$\Theta(f) = \hat{\mathcal{A}}_{ac}(f)$$
$$\mathcal{A}_{ac}(f) = -\hat{\Theta}(f). \qquad (85)$$

Cascade the above filter with impulse response $g(t)$ and transfer function $\mathcal{G}(f)$, with an ideal bandpass filter of width $2B$ centered around f_0; denote the overall impulse response and transfer function by $h(t)$ and $\mathcal{H}(f)$:

$$\mathcal{H}(f) \equiv \begin{cases} \mathcal{G}(f), & |f-f_0| < B, \\ 0, & |f-f_0| > B, \end{cases} \qquad B < f_0. \qquad (86)$$

Let $r(t)$ denote the envelope [12] of the overall impulse response $h(t)$:

$$r(t) \equiv |h(t) + j\hat{h}(t)| \qquad (87)$$

$\hat{h}(t)$ being the Hilbert transform of $h(t)$. Then [12]

$$r^2(t) = 4\int_{-2B}^{2B}\left\{\int_{f_0-B-(f-|f|)/2}^{f_0+B-(f+|f|)/2} \mathcal{G}(a+f)\mathcal{G}^*(a)da\right\} \cdot \exp(j2\pi ft)df. \qquad (88)$$

We now assume that the transfer function $\mathcal{G}(f)$ is wide-sense stationary; i.e., its mean $\langle\mathcal{G}(f)\rangle$ is independent of f, and its covariance $\langle\mathcal{G}(f+\nu)\mathcal{G}^*(f)\rangle$ depends only on ν and not on f. The fact that $\langle\mathcal{G}(f)\rangle$ is independent of f, together with (80) and (81), yields

$$\langle\mathcal{A}(f)\rangle \equiv \langle\mathcal{A}\rangle$$
$$\langle\Theta(f)\rangle = 0$$
$$\langle\mathcal{G}(f)\rangle = 1 - \langle\mathcal{A}\rangle. \qquad (89)$$

We denote the covariance by

$$\mathcal{R}(\nu) \equiv \langle\mathcal{G}(f+\nu)\mathcal{G}^*(f)\rangle, \qquad \mathcal{R}(\nu) = \mathcal{R}^*(-\nu). \qquad (90)$$

Define

$$\mathcal{P}(t) \equiv \int_{-\infty}^{\infty} \mathcal{R}(\nu)\exp(-j2\pi t\nu)d\nu, \qquad \mathcal{P}(t) = \mathcal{P}^*(t). \qquad (91)$$

$\mathcal{P}(t)$ is the spectral density of the transfer function $\mathcal{G}(f)$; it is real, but not symmetric. Assume that $\mathcal{P}(t)$ contains no δ-function components except possibly at the origin $t=0$, i.e., $\propto \delta(t)$; consequently $\mathcal{G}(f)$ contains no periodic components, but may contain a dc component. Then

$$\mathcal{R}(\infty) = \mathcal{R}(-\infty) = \langle|\overline{\mathcal{G}(f)}|^2\rangle$$
$$= \langle(1-\bar{\mathcal{A}})^2\rangle = 1 - 2\langle\mathcal{A}\rangle + \langle\bar{\mathcal{A}}^2\rangle \qquad (92)$$

$\bar{\mathcal{A}}$ and $\langle\mathcal{A}\rangle$ being given by (82) and (89).[15] Using (92), define $\mathcal{R}^{ac}$ and $\mathcal{P}^{ac}$ by

$$\mathcal{R}(\nu) \equiv (1 - 2\langle\mathcal{A}\rangle + \langle\bar{\mathcal{A}}^2\rangle) + \mathcal{R}^{ac}(\nu)$$
$$\mathcal{R}^{ac}(\infty) = 0$$
$$\mathcal{P}(t) \equiv (1 - 2\langle\mathcal{A}\rangle + \langle\bar{\mathcal{A}}^2\rangle)\delta(t) + \mathcal{P}^{ac}(t). \qquad (93)$$

Further, define

$$\mathcal{G}_{ac}(f) \equiv -\mathcal{A}_{ac}(f) + j\Theta(f) \qquad (94)$$

where $\mathcal{A}_{ac}$ is defined by (83). Then

$$\mathcal{R}^{ac}(\nu) = \langle\mathcal{G}_{ac}(f+\nu)\mathcal{G}_{ac}^*(f)\rangle. \qquad (95)$$

$\mathcal{P}^{ac}(t)$ contains no δ functions; it is the Fourier transform of $\mathcal{R}^{ac}(\nu)$ via (91), and is the spectral density of $\mathcal{G}_{ac}(f)$. From (85), $\mathcal{P}^{ac}(-t)$ [and $\mathcal{P}(-t)$] are causal, and the real and imaginary parts of the covariance $\mathcal{R}^{ac}(\nu)$ are Hilbert transforms [12]:

$$\mathcal{P}^{ac}(t) = 0, \qquad t > 0 \qquad (96)$$
$$\mathrm{Im}\,\mathcal{R}^{ac}(\nu) = -\widehat{\mathrm{Re}\,\mathcal{R}^{ac}}(\nu)$$
$$\mathrm{Re}\,\mathcal{R}^{ac}(\nu) = \widehat{\mathrm{Im}\,\mathcal{R}^{ac}}(\nu). \qquad (97)$$

Finally, the second-order statistics of $\mathcal{A}^{ac}$ (or $\mathcal{A}$) and Θ are easily expressed in terms of $\mathcal{R}^{ac}$ and $\mathcal{P}^{ac}$. Define the co- and cross variances of $\mathcal{A}^{ac}$ and Θ as follows:

$$\mathcal{R}_{\mathcal{A}_{ac}}(\nu) \equiv \langle\mathcal{A}_{ac}(f+\nu)\mathcal{A}_{ac}(f)\rangle$$
$$\mathcal{R}_{\Theta}(\nu) \equiv \langle\Theta(f+\nu)\Theta(f)\rangle$$
$$\mathcal{R}_{\mathcal{A}_{ac}\Theta}(\nu) \equiv \langle\mathcal{A}_{ac}(f+\nu)\Theta(f)\rangle$$
$$\mathcal{R}_{\Theta\mathcal{A}_{ac}}(\nu) \equiv \langle\Theta(f+\nu)\mathcal{A}_{ac}(f)\rangle. \qquad (98)$$

The self- and cross-spectral densities are the Fourier transforms of these four quantities given by (91) with corresponding subscripts on $\mathcal{P}$ and $\mathcal{R}$. Then [12]

$$\mathcal{R}_{\mathcal{A}_{ac}}(\nu) = \mathcal{R}_{\Theta}(\nu) = \tfrac{1}{2}\,\mathrm{Re}\,\mathcal{R}^{ac}(\nu) \qquad (99)$$
$$\mathcal{R}_{\mathcal{A}_{ac}\Theta}(\nu) = -\mathcal{R}_{\Theta\mathcal{A}_{ac}}(\nu) = \tfrac{1}{2}\,\mathrm{Im}\,\mathcal{R}^{ac}(\nu) \qquad (100)$$
$$\mathcal{P}_{\mathcal{A}_{ac}}(t) = \mathcal{P}_{\Theta}(t) = \tfrac{1}{4}[\mathcal{P}^{ac}(-t) + \mathcal{P}^{ac}(t)] \qquad (101)$$
$$\mathcal{P}_{\mathcal{A}_{ac}\Theta}(t) = -\mathcal{P}_{\Theta\mathcal{A}_{ac}}(t) = \frac{j}{4}[\mathcal{P}^{ac}(-t) - \mathcal{P}^{ac}(t)]. \qquad (102)$$

The discussion of (89)–(102) characterizes the second-order statistics of a wide-sense stationary transfer function in the frequency domain. We desire to characterize such a random filter in the time domain. To do so we cascade the random filter with the ideal bandpass filter described just above (86). Equation (88) gives the envelope of the overall impulse response. We take the expected value of both sides of (88) and use (90),[16] obtaining the average squared envelope of the impulse re-

[15] Note that $\langle\bar{\mathcal{A}}^2\rangle$ is *not* in general equal to $\overline{\langle\mathcal{A}\rangle}^2 = \langle\bar{\mathcal{A}}\rangle^2$; in the special case where these quantities are equal, $\mathcal{A}(f)$ and $\mathcal{G}(f)$ are deterministic quantities [see (112)–(115)].

[16] $\mathcal{G}(f)$ need now be only locally wide-sense stationary over the band $|f-f_0|<B$, rather than over the infinite frequency band as assumed in (90).

sponse as

$$\langle i^2(t)\rangle = 8B\int_{-2B}^{2B}\left(1+\frac{|f|}{2B}\right)\mathcal{R}(f)\exp(j2\pi ft)df. \quad (103)$$

From (91) [17]

$$\langle i^2(t)\rangle = 8B\left\{2B\left(\frac{\sin 2\pi Bt}{2\pi Bt}\right)^2\right\} * \mathcal{P}(-t) \quad (104)$$

$*$ again representing the convolution operator. From (93)

$$\langle i^2(t)\rangle = 8B(1-2\langle \mathcal{a}\rangle + \langle\bar{\mathcal{a}}^2\rangle)\cdot\left\{2B\left(\frac{\sin 2\pi Bt}{2\pi Bt}\right)^2\right\} + 8B\left\{2B\left(\frac{\sin 2\pi Bt}{2\pi Bt}\right)^2\right\} * \mathcal{P}^{ac}(-t). \quad (105)$$

The first line of (105) represents the ideal response obtained in the absence of statistical fluctuations in the transfer function $\mathcal{G}(f)$; the second line represents the echoes due to transfer-function fluctuations. The quantity in $\{\ \}$ in (105) has unit area; thus for large enough B [i.e., half-bandwidth of the ideal bandpass filter cascaded with $\mathcal{G}(f)$] it resembles a unit impulse function. We assume the following.

1) B is large compared to the correlation bandwidth of $\mathcal{G}(f)$; i.e., we examine a piece of $\mathcal{G}(f)$ containing many fluctuations. Then $\{\ \}$ in the second line of (105) is narrow compared to $\mathcal{P}^{ac}(-t)$ and may be dropped.

2) $\{\ \}$ in the first line of (105) is narrow compared to any pulse we wish to transmit over the channel in any case where the first line is significant; consequently, we replace it by a unit impulse.

Under these conditions, (105) becomes

$$\begin{aligned}\langle i^2(t)\rangle &\approx 8B[(1-2\langle\mathcal{a}\rangle+\langle\bar{\mathcal{a}}^2\rangle)\delta(t)+\mathcal{P}^{ac}(-t)]\\ &= 8B\mathcal{P}(-t), \qquad B \text{ large.}\end{aligned} \quad (106)$$

For these limiting conditions $\mathcal{P}(-t)$ approximates the normalized *expected squared envelope of the impulse response* of the random transfer function $\mathcal{G}(f)$; we abbreviate the italic phrase by *pulse response* throughout the remainder of this paper. Thus $\mathcal{P}(t)$ is the spectral density of $\mathcal{G}(f)$ (91) and $\mathcal{P}(-t)$ is the (normalized) *pulse response*; $\mathcal{P}^{ac}(-t)$ represents the average echo power produced by the transmission fluctuations.

The relations of this section (75)–(106) apply directly to the solutions of the coupled line equations described in Sections I–IX by the following substitutions:

$$\begin{gathered}\wp \to g \quad \mathcal{P}\to P \quad t\to\tau \quad f\to -\frac{\Delta\beta z}{2\pi} \quad \nu\to -\frac{\sigma z}{2\pi}\\ \mathcal{G}(f)\to G_0(\Delta\beta) \quad \mathcal{A}(f)\to A(\Delta\beta) \quad \Theta(f)\to\theta(\Delta\beta)\\ \mathcal{R}(\nu)\to R_0(\sigma).\end{gathered} \quad (107)$$

This is so because the solutions to (1), the coupled line equations, with the initial conditions of (2), satisfy the assumptions of this section.

1) The normalized impulse response for the signal mode in the coupled line equations is real and causal in general (15), (28).

2) The signal transfer function of the coupled line equations for white coupling $c(z)$ with independent successive values is wide-sense stationary over the infinite range $-\infty<\Delta\beta<\infty$ [(56) and the first relation of (70)].

Thus $P(\tau)$, the spectral density of the normalized signal transfer function of the coupled line equations for white coupling, is given by

$$P(\tau) \equiv \int_{-\infty}^{\infty} R_0(\sigma)\exp(j\tau\sigma z)d\left(\frac{\sigma z}{2\pi}\right) \quad (108)$$

with $R_0(\sigma)$ given by (70). $P(-\tau)$ is the normalized *pulse response* for the signal mode. Substituting (70) into (108), calculations outlined in Appendix III yield

$$P(\tau) = \exp(-S_0 z)\cdot\delta(\tau) + P^{ac}(\tau)$$

$$P^{ac}(\tau) = \begin{cases} S_0 z\cdot\exp(-S_0 z)\exp(-2\Delta\alpha z\tau)\sqrt{\dfrac{1+\tau}{-\tau}} \\ \quad\cdot I_1(2S_0 z\sqrt{-\tau(1+\tau)}), & -1<\tau<0\\ 0, & \text{otherwise}\end{cases} \quad (109)$$

where I_1 represents a modified Bessel function of first order. The behavior of $R_0(\sigma)$ of (70) and $P(\tau)$ of (109)—the two main results of this paper—is discussed for a number of cases of interest in Section XI.[17]

Consider again the dispersionless case discussed in Sections III and V (30)–(33), and Appendix I (147)–(149), in which $\Delta\beta$ is strictly proportional to f over the infinite frequency range. The *pulse response* $\mathcal{P}(-t)$ in the actual time domain is given in terms of (109) by

$$\begin{aligned}\mathcal{P}(t) &= \frac{1}{T}P\left(\frac{t}{T}\right) = \exp(-S_0 z)\cdot\delta(t) + \mathcal{P}^{ac}(t)\\ \mathcal{P}^{ac}(t) &= \frac{1}{T}\cdot P^{ac}\left(\frac{t}{T}\right)\end{aligned} \quad (110)$$

where from (33)

$$T \equiv z\left(\frac{1}{v_1}-\frac{1}{v_0}\right) \quad (111)$$

is again the delay between signal and spurious modes for a length z of transmission line. The *pulse response* $\mathcal{P}(-t)$ (expected squared envelope of the impulse response) and the impulse response $\wp(t)$ (32) are both time limited to the interval $0\leq t<T$.

XI. Examples

We finally illustrate the general behavior of the signal mode transmission statistics of a two-mode transmission system for white coupling with independent successive values (e.g., white Gaussian coupling).

[17] The Fourier transform of $R_1(\sigma)$ of (70) is similarly obtained from (162).

The normalized transfer function $G_0(\Delta\beta)$ is wide-sense stationary, since $\langle G_0(\Delta\beta)\rangle$ (Eq. 56), and $R_0(\sigma)=\langle G_0(\Delta\beta+\sigma)G_0^*(\Delta\beta)\rangle$ (Eq. 70), are independent of $\Delta\beta$. It is obvious that $G_0(\Delta\beta)$ is *not* strictly stationary; for example, $G_0(-\Delta\beta)=G_0^*(\Delta\beta)$ (Eq. 28), $\langle G_0^2(\sigma/2)\rangle=R_0(\sigma)$ (Eq. 70).

Next, noting that $R_0(\infty)$, the coefficient of the $\delta(\tau)$ component of $P(\tau)$, gives the dc power of the transfer function [(92), (93), and (107)], we have from (70) and (109)

$$R_0(\infty) = \langle |\overline{G_0(\Delta\beta)}|^2\rangle = \langle[1-\overline{A(\Delta\beta)}]^2\rangle = \exp(-S_0z) \tag{112}$$

for all $\Delta\alpha\leq 0$. From (56)

$$\langle G_0(\Delta\beta)\rangle^2 = \langle 1 - A(\Delta\beta)\rangle^2 = \exp(-S_0z). \tag{113}$$

Equation (113) is independent of $\Delta\beta$; interchanging $\langle\ \rangle$ and $\overline{}$,

$$\langle G_0(\Delta\beta)\rangle = \langle\overline{G_0(\Delta\beta)}\rangle$$
$$\langle A(\Delta\beta)\rangle = \langle\overline{A(\Delta\beta)}\rangle. \tag{114}$$

Since (112) and (113) are equal,

$$\langle\overline{A(\Delta\beta)}\rangle^2 = \langle\overline{A(\Delta\beta)}^2\rangle$$
$$\langle\overline{G(\Delta\beta)}\rangle^2 = \langle\overline{G(\Delta\beta)}^2\rangle. \tag{115}$$

Therefore, $\overline{A(\Delta\beta)}$ and $\overline{G(\Delta\beta)}$ are deterministic; i.e., the dc component of the transfer function is *not* a random variable. This result is not true for all random media problems.

Next, let us consider the ac power of the transfer function, i.e., the mean square value of the ac fluctuations about its dc component. The total power is

$$R_0(0) = \langle |G_0(\Delta\beta)|^2\rangle. \tag{116}$$

The dc power is given by (112). Recalling that [(80)–(84), (94), and (107)]

$$G_{0\,\mathrm{ac}}(\Delta\beta) \equiv G_0(\Delta\beta) - \overline{G_0(\Delta\beta)} \tag{117}$$

the ac power is

$$R_0(0) - R_0(\infty) = \langle |G_{0\,\mathrm{ac}}(\Delta\beta)|^2\rangle. \tag{118}$$

From (70) and (109),

$$R_0(0) - R_0(\infty) = \exp(-S_0z)\exp(-|\Delta\alpha|z) \cdot\left(\cosh(z\sqrt{S_0^2+\Delta\alpha^2}) + |\Delta\alpha|\frac{\sinh(z\sqrt{S_0^2+\Delta\alpha^2})}{\sqrt{S_0^2+\Delta\alpha^2}}\right) - \exp(-S_0z), \qquad \Delta\alpha\leq 0. \tag{119}$$

The general behavior of (118), (119), and (112) is shown by the limiting cases of Table I.

The first row of Table I, $S_0=0$, states the obvious fact that if there is no coupling, there are no transmission fluctuations, and therefore the transmission is ideal.

TABLE I
TRANSFER FUNCTION STATISTICS $\Delta\alpha\leq 0$

	AC Power	DC Power	AC Power / DC Power
	$R_0(0) - R_0(\infty)$	$R_0(\infty)$	$\dfrac{R_0(0) - R_0(\infty)}{R_0(\infty)}$
$S_0 = 0$	0	1	0
$\Delta\alpha = 0$	$\dfrac{(1-\exp(-S_0z))^2}{2}$	$\exp(-S_0z)$	$2\sinh^2\dfrac{S_0z}{2}$
$\Delta\alpha = -\infty$	0	$\exp(-S_0z)$	0
$S_0 = \infty$	$\dfrac{\exp(-\lvert\Delta\alpha\rvert z)}{2}$	0	∞

The second row treats the lossless case $\Delta\alpha=0$. As S_0z increases from 0 to ∞, the ac power increases from 0 to $\frac{1}{2}$, and the dc power decreases exponentially from 1 to 0. For $S_0z\gg 1$ the transfer function fluctuations are much greater than the average transfer function. Naïvely one might think this implies poor transmission, but this is not so. A rough idea of how this might come about can be seen by considering random transfer functions of the form

$$\mathcal{G}(f) = \exp(j\theta),\ +\theta \text{ uniformly distributed from } 0\rightarrow 2\pi. \tag{120}$$

Clearly, $\langle\mathcal{G}\rangle=\overline{\mathcal{G}}=0$, i.e., the average transfer function is zero, while the transfer function fluctuation over the ensemble of guides has unit power; nevertheless, such a transfer function introduces no distortion, since the transfer function for each guide is constant with frequency. The discussion of $R_0(\tau)$ and $P(\tau)$ below demonstrates the surprising fact that the transmission distortion decreases as the coupling S_0z increases for large enough S_0z; the transmission distortion is of course zero for $S_0z=0$, and approaches zero as $S_0z\rightarrow\infty$.

The third row of Table I demonstrates that, for fixed length z, making the spurious-mode heat loss $|\Delta\alpha|$ large enough will smooth out the transmission fluctuations. To illustrate, consider the case $S_0z\gg 1$, for which (by the second line of Table I) the percentage transmission fluctuations over the ensemble are large for $\Delta\alpha=0$. From (118), (119), and (112):

$$\frac{\text{ac power}}{\text{dc power}} = \frac{R_0(0)-R_0(\infty)}{R_0(\infty)} = \exp\left(\frac{S_0z}{2}\cdot\frac{S_0}{|\Delta\alpha|}\right) - 1,$$
$$\Delta\alpha\leq 0,\quad S_0z\gg 1,\quad \frac{S_0}{|\Delta\alpha|}\ll 1,\quad S_0z\left(\frac{S_0}{|\Delta\alpha|}\right)^3\ll 8. \tag{121}$$

Therefore, for the transfer function fluctuations to be small compared to the average transfer function we must have

$$\frac{|\Delta\alpha|}{S_0} \gg \frac{S_0z}{2}\,. \tag{122}$$

The above discussion of this section has been confined to transfer-function statistics at a single frequency [$R_0(0)$ and $R_0(\infty)$]. It is clear from the example of (120) that this is not sufficient to characterize transmission distortion. To do so we must study the functional dependence of $R_0(\sigma)$ and $P(\tau)$, (70) and (109), for all values of their arguments, thus obtaining statistics of the frequency variation of the transfer function and of the impulse response.

We begin with the lossless case $\Delta\alpha=0$ for simplicity. Note from Section IV, 3) that the lossy case $\Delta\alpha<0$ is a straightforward extension. Then the following three values of the transfer function covariance (70) are of interest:

$$\left.\begin{aligned} R_0(0) &= \frac{1+\exp(-2S_0z)}{2} \\ R_0(2S_0) &= \exp(-S_0z)\exp(+jS_0z)([1-jS_0z]) \\ R_0(\infty) &= \exp(-S_0z) \end{aligned}\right\} \quad \Delta\alpha = 0. \tag{123}$$

We consider separately the perturbation case $S_0z\ll 1$ and the complementary case $S_0z\gg 1$, in which perturbation theory is grossly violated.

For the perturbation case, (123) yields to second order:

$$\left.\begin{aligned} R_0(0) &\approx 1 - S_0z + (S_0z)^2 \\ R_0(2S_0) &\approx 1 - S_0z + (S_0z)^2 \\ R_0(\infty) &\approx 1 - S_0z + \frac{(S_0z)^2}{2} \end{aligned}\right\} \quad S_0z \ll 1, \quad \Delta\alpha = 0. \tag{124}$$

It is obvious that no significant change in $R_0(\sigma)$ of (70) occurs for $\sigma<2S_0$. Rewriting (70) in a form appropriate to the region of interest, $\sigma\gg 2S_0$, we have exactly

$$R_0(\sigma) = \exp(-S_0z)\left\{\exp\left[j\frac{\sigma}{2}z\left(1-\sqrt{1-\left(\frac{2S_0}{\sigma}\right)^2}\right)\right] - j\frac{1-\sqrt{1-\left(\frac{2S_0}{\sigma}\right)^2}}{\sqrt{1-\left(\frac{2S_0}{\sigma}\right)^2}}\exp\left(j\frac{\sigma}{2}z\right) \cdot \sin\left(\frac{\sigma}{2}z\sqrt{1-\left(\frac{2S_0}{\sigma}\right)^2}\right)\right\}, \quad \Delta\alpha = 0. \tag{125}$$

In the range of interest, approximating the radicals yields

$$R_0(\sigma) \approx \exp(-S_0z)\left[1+\frac{(S_0z)^2}{2}\left\{\frac{\sin\frac{\sigma z}{2}}{\frac{\sigma z}{2}}\right\}^2 + j(S_0z)^2\frac{1}{\sigma z}\cdot\left(1-\frac{\sin\sigma z}{\sigma z}\right)\right], \quad S_0z \ll 1, \quad \Delta\alpha = 0. \tag{126}$$

Likewise, approximating the modified Bessel function I_1 in (109) by half its argument yields for the (normalized) *pulse response*[18]

$$\begin{aligned} P(-\tau) &= \exp(-S_0z)\cdot\delta(\tau) + P^{ac}(-\tau), \\ P^{ac}(-\tau) &\approx \exp(-S_0z)\cdot(S_0z)^2(1-\tau), \quad 0 \le \tau \le 1, \\ &\qquad S_0z \ll 1, \quad \Delta\alpha = 0. \end{aligned} \tag{127}$$

Equations (126) and (127) are precise Fourier transforms [via (108)].

The above agrees with prior results of perturbation theory. For example, from (113) we have exactly

$$\langle A\rangle = 1 - \exp(-S_0z/2), \quad \Delta\alpha \le 0. \tag{128}$$

In the perturbation case this becomes

$$\langle A\rangle \approx \frac{S_0z}{2}, \quad S_0z \ll 1, \quad \Delta\alpha \le 0 \tag{129}$$

which agrees with [5, eq. (300)] for $\Delta\alpha=0$. From (101), (107), and the second line of (127) with the factor $\exp(-S_0z)$ dropped,

$$P_{A_{ac}}(\tau) = \frac{(S_0z)^2}{4}(1-|\tau|), \quad |\tau| \le 1, \quad S_0z \ll 1, \quad \Delta\alpha = 0 \tag{130}$$

gives the spectrum of the ac component of the loss A. This agrees with [5, eq. (303)]. Equation (127) for $P^{ac}(-\tau)$ shows the average echo power is triangular. In perturbation theory echoes arise from all pairs of mode converters [the $n=1$ term of (153)]. The triangular distribution arises (for $\Delta\alpha=0$) because there are more pairs close together ($\tau\backsim 0$) than far apart ($\tau\backsim 1$) and none separated by $\tau\ge 1$.

Next, consider long lines with large coupling $S_0z\gg 1$, where perturbation theory fails. We again initially restrict our attention to the lossless case $\Delta\alpha=0$. From (123):

$$\left.\begin{aligned} R_0(0) &\approx \tfrac{1}{2} \\ |R_0(2S_0)| &\approx S_0z\cdot\exp(-S_0z), \\ R_0(\infty) &= \exp(-S_0z) \end{aligned}\right\} \quad S_0z \gg 1, \quad \Delta\alpha = 0. \tag{131}$$

Here the significant range of σ for $R_0(\sigma)$ is $\sigma\ll 2S_0$ in contrast to the perturbation case above. The following approximation to (70) is suitable in this range:

$$R_0(\sigma) \approx \frac{1}{2}\exp\left(j\frac{\sigma z}{2}\right)\exp\left(-\frac{\sigma^2 z}{8S_0}\right), \quad S_0z \gg 1, \quad \Delta\alpha = 0. \tag{132}$$

The corresponding *pulse response* may be obtained from (109) by observing that for $S_0z\gg 1$ the asymptotic approximation $I_1(x)\sim\exp(x)/\sqrt{2\pi x}$ is appropriate except near $\tau\backsim 0$ or $\tau\backsim -1$, and that $P^{ac}(\tau)$ is negligible except near $\tau\backsim -\frac{1}{2}$.

[18] Recall *pulse response* is the expected squared envelope of the impulse response, as defined following (106).

$$P(-\tau) \approx P^{ac}(-\tau) \approx \sqrt{\frac{S_0 z}{2\pi}} \exp\left(-2S_0 z\left(\tau - \frac{1}{2}\right)^2\right),$$
$$0 < \tau < 1, \quad S_0 z \gg 1, \quad \Delta\alpha = 0. \quad (133)^{19}$$

Both the covariance $R_0(\sigma)$ and $P^{ac}(\tau)$ are approximately Gaussian for $S_0 z \gg 1$; the approximations of (132) and (133) are precise Fourier transforms [via (108)].

Equation (133) represents a Gaussian pulse centered on $\tau = \frac{1}{2}$, of width proportional to

$$\Delta\tau = \sqrt{\frac{2}{S_0 z}}, \qquad S_0 z \gg 1, \quad \Delta\alpha = 0. \qquad (134)$$

For the dispersionless case, (110) and (111) give the *pulse response* in the actual time domain, again Gaussian centered on $t = T/2$, of width

$$\Delta t = T\sqrt{\frac{2}{S_0 z}}, \qquad S_0 z \gg 1, \quad \Delta\alpha = 0. \qquad (135)$$

T is again the delay between signal and spurious modes for a length z of transmission line,

$$T \equiv z\left(\frac{1}{v_1} - \frac{1}{v_0}\right) \qquad (136)$$

v_0 and v_1 being the (frequency-independent) mode velocities [Appendix I, 1)]. These results apply to the expected square envelope of the impulse response; the impulse response itself and its envelope $\imath(t)$ are random functions. From the Chebyshev inequality and (106), the tails of the probability density for $\imath(t)$ are bounded by

$$\text{probability}\,[\imath(t) \geq \epsilon] \leq \frac{8B \cdot \mathcal{P}(-t)}{\epsilon^2} \cdot \qquad (137)$$

Therefore, the (random) impulse response and its envelope $\imath(t)$ are localized to the same time interval as the *pulse response* $\mathcal{P}(-t)$, and consequently (135) gives a good estimate for the duration of the impulse response.

Equation (135) gives the remarkable result that the duration of the impulse response *decreases* as the coupling increases (i.e., as the spectral density S_0 of the coupling increases). In colloquial language, the worse you make the transmission line (i.e., the larger its random geometric imperfections), the better it works. This behavior was first pointed out by Personick [19], using a different model and analysis. We note from the second row of Table I that the ac power of the transfer function far exceeds the dc power in the present case ($S_0 z \gg 1$, $\Delta\alpha = 0$). It is helpful to consider the limiting case as $S_0 \to \infty$ and the *pulse response* becomes

$$\lim_{S_0 \to \infty} P(-\tau) \to \tfrac{1}{2}\,\delta(\tau - \tfrac{1}{2}), \qquad \Delta\alpha = 0. \qquad (138)$$

[19] The δ-function component of $P(-\tau)$ [see (109)] vanishes because $S_0 z \gg 1$.

From (132) the transfer-function covariance becomes

$$\lim_{S_0 \to \infty} R_0(\sigma) \to \frac{1}{2} \exp\left(j\frac{\sigma z}{2}\right), \qquad \Delta\alpha = 0 \qquad (139)$$

corresponding to sample functions

$$\lim_{S_0 \to \infty} G_0(\Delta\beta) \to \frac{K}{\sqrt{2}} \exp\left(j\frac{\Delta\beta z}{2}\right) \exp(j\theta), \qquad \Delta\alpha = 0,$$
$$\langle \exp(j\theta) \rangle = 0,$$
$$\langle K^2 \rangle = 1. \qquad (140)$$

We might guess from the physics of the problem that θ is uniformly distributed from $0 \to 2\pi$, although this does not follow from this work. The real parameter K is a random variable, proportional to the magnitude of the transfer function G_0 or to the magnitude of the impulse response; since only its mean square value is known, this work tells nothing about the transfer function or impulse-response magnitude fluctuations from guide to guide. These limiting transfer functions have dc power $= 0$ and ac power $= 0.5$ (averaged over the ensemble of random guides), as in row two of Table I for $S_0 \to \infty$ or row four for $\Delta\alpha = 0$. Thus, in the lossless case ($\Delta\alpha = 0$) for large coupling ($S_0 z \gg 1$) the signal transfer function G_0 has approximately constant magnitude (which varies from guide to guide) and linear phase, corresponding to a narrow impulse response centered around the average of the transit times for the two modes. A physical explanation for this average delay is that the pulse travels approximately equally in both modes for large $S_0 z$; the significant terms in the time-domain description associated with (35) have large n, corresponding to a large number of transitions between modes.

We have seen that for $\Delta\alpha = 0$ the average echo power $P^{ac}(-\tau)$ is approximately triangular in the perturbation case (127), $S_0 z \ll 1$, and approximately Gaussian for large coupling (133), $S_0 z \gg 1$. Fig. 3 shows the exact behavior of this function for a range of $S_0 z$ spanning these limiting types of behavior, obtained from (109). Note that $S_0 z$ does not become large enough in this figure for the approximation of (133) to be well satisfied in every respect; in particular, for $S_0 z = 10$ the peak of the Gaussian-shaped curve is located at $\tau = 0.45$ in Fig. 3, rather than at $\tau = 0.5$ in (133); an improved Gaussian approximation, (143) below, gives excellent agreement with the exact results even for $S_0 z$ as small as 3.

Finally, we include the effect of differential loss $\Delta\alpha < 0$. From (128) and (115) the average loss is independent of $\Delta\alpha$. From (109) the general *pulse response* (for some negative $\Delta\alpha$) is given in terms of the $\Delta\alpha = 0$ *pulse response* by

$$P^{ac}(-\tau) = \exp(-2|\Delta\alpha| z\tau) \cdot [P^{ac}(-\tau)]_{\Delta\alpha=0},$$
$$\Delta\alpha \leq 0 \quad (141)$$

similar to (19). Consequently, $R_0(\sigma)$ of (70) and its real

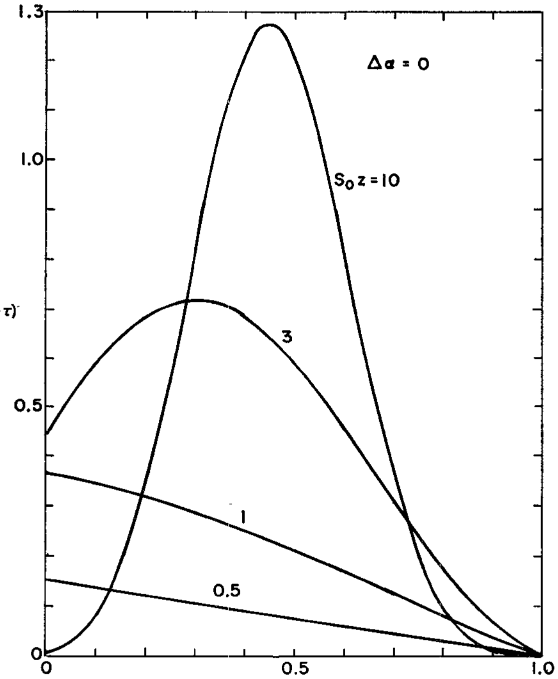

Fig. 3. Average echo power (*pulse response* with delta-function component $\exp(-S_0 z)\cdot\delta(\tau)$ omitted) with coupling as a parameter.

and imaginary parts for $\Delta\alpha<0$ are related to the corresponding quantities for $\Delta\alpha=0$ by expressions similar to (20)–(22) by the substitutions

$$G_0 \to R_0 \qquad \Delta\beta \to \sigma$$
$$A \to -\operatorname{Re} R_0 \qquad \theta \to \operatorname{Im} R_0$$
$$\Delta\alpha \to 2\Delta\alpha \tag{142}$$

[20]

As a single example, let us find the approximate *pulse response* for long lines $S_0 z \gg 1$ as a function of $\Delta\alpha$. Using the two-term asymptotic approximation $I_1(x) \sim (\exp(x)/\sqrt{2\pi x})(1-[3/8x])$ in (109), we take a three-term Taylor expansion of $\ln P(-\tau)$ about $\tau=\frac{1}{2}$ [including terms up to $(\tau-\frac{1}{2})^2$] to yield

$$P(-\tau) \approx P^{ac}(-\tau) \approx \sqrt{\frac{S_0 z}{2\pi}}\left(1-\frac{3}{8S_0 z}\right)\exp(-|\Delta\alpha| z)$$
$$\cdot\exp\left\{\frac{(1+|\Delta\alpha| z)^2}{2S_0 z - 1 + \dfrac{3}{4S_0 z}\Big/\left(1-\dfrac{3}{8S_0 z}\right)}\right\}$$
$$\cdot\exp\left\{-\left[2S_0 z - 1 + \frac{3}{4S_0 z}\Big/\left(1-\frac{3}{8S_0 z}\right)\right]\right.$$
$$\left.\cdot\left[\tau - 0.5 + \frac{1+|\Delta\alpha| z}{2S_0 z - 1 + \dfrac{3}{4S_0 z}\Big/\left(1-\dfrac{3}{8S_0 z}\right)}\right]^2\right\}$$
$$\Delta\alpha \le 0, \qquad S_0 \gg 1, \qquad 0<\tau<1 \tag{143}$$

[20] These substitutions apply also to other results of Section IV, overlapping portions of Section X.

which may be further simplified to

$$P(-\tau) \approx P^{ac}(-\tau) \approx \sqrt{\frac{S_0 z}{2\pi}}\exp(-|\Delta\alpha| z)$$
$$\cdot\exp\left[\frac{(1+|\Delta\alpha| z)^2}{2S_0 z - 1}\right]$$
$$\cdot\exp\left[-(2S_0 z - 1)\left(\tau - 0.5 + \frac{1+|\Delta\alpha| z}{2S_0 z - 1}\right)^2\right],$$
$$\Delta\alpha \le 0, \qquad S_0 z \gg 1, \qquad 0<\tau<1 \tag{144a}$$

for most purposes. For $\Delta\alpha=0$ this approximation is an improvement over that of (133), which predicts the peak of the *pulse response* occurs at $\tau=0.5$. Equation (144a) predicts the peak for $S_0 z=10$ will occur at $\tau\approx 0.45$, which is in close agreement with the exact result of Fig. 3. For $\Delta\alpha<0$ the peak moves toward $\tau=0$, maintaining constant width. This approximation clearly fails when the predicted peak occurs at $\tau=0$; consequently, the requirement

$$\frac{|\Delta\alpha|}{S_0} \ll 1 \tag{144b}$$

must also be satisfied for the approximation of (144a) [or (143)] to hold. Within the range of (144b), the magnitude of the peak decreases as $\Delta\alpha$ increases. Consequently, within the region of (144b) increasing the spurious mode loss $|\Delta\alpha|$ simply increases the overall signal–signal loss, without otherwise altering the transmission. This is physically reasonable, since for $\Delta\alpha=0$ we have seen [following (140)] that the pulse may be regarded as equally divided between the two modes, and so must also share their heat losses. For much larger $|\Delta\alpha|$, the τ dependence of (109) is due almost entirely to the $\exp(-2\Delta\alpha z\tau)$ factor; we have

$$P^{ac}(-\tau) \approx (S_0 z)^2 \exp(-S_0 z)\exp[-(2|\Delta\alpha| z+1)\tau],$$
$$\frac{|\Delta\alpha|}{S_0} \gg 1, \qquad |\Delta\alpha| z \gg 1 \tag{145}$$

for $S_0 z$ large or small. The area of $P^{ac}(-\tau)$ equals the ac power of the transfer-function fluctuations; from (145)

$$\int_0^1 P^{ac}(-\tau)d\tau \approx \frac{(S_0 z)^2\exp(-S_0 z)}{2|\Delta\alpha| z+1}, \qquad \frac{|\Delta\alpha|}{S_0} \gg 1,$$
$$|\Delta\alpha| z \gg 1. \tag{146}$$

When (146) is small compared to $\exp(-S_0 z)$, the area of the δ-function component of (109), the ac power will be small compared to the dc power, and consequently the transfer-function fluctuations will smooth out; this condition is identical to that previously given in (122) in the region where both apply.

XII. Discussion

The present methods permit exact treatment of transmission statistics in either the time or frequency domains for multimode media for white coupling with independent successive values (e.g., white Gaussian coupling), and forward signal and spurious modes. The signal distortion in long lines may thus be studied exactly. Many of the assumptions of the present treatment are easily removed at the cost of more complicated, but straightforward, calculations. We can consider systems with more forward modes, outputs in several modes, and we can compute higher order transmission statistics by similar methods.

In contrast, other assumptions do not appear easily removed. We cannot treat nonwhite coupling spectra rigorously beyond guessing that low-pass spectra of Fig. 2 subject to the restriction of (53) will exhibit similar behavior to that computed here for strictly white coupling spectra. We cannot treat multimode systems with backward waves, except in the special case of two modes (one forward, one backward) [10], and then only by compromising in calculating loss statistics (rather than gain statistics, as in this work).

For large enough coupling S_0z, the transmission distortion will be small. Then for zero differential loss, $\Delta\alpha=\alpha_0-\alpha_1=0$, the signal loss is $\frac{1}{2}\exp(-2\alpha_0 z)$. As the differential loss $|\Delta\alpha|$ increases, but remains small compared to the coupling, $|\Delta\alpha|\ll S_0$, the signal loss is $\frac{1}{2}\exp[-(\alpha_0+\alpha_1)z]=\frac{1}{2}\exp(-2\alpha_0 z)\cdot\exp(-|\Delta\alpha|z)$. Finally, for large differential loss compared to the coupling, $|\Delta\alpha|\gg S_0$, the signal distortion decreases, and the signal loss becomes $\exp(-S_0z)$ for $|\Delta\alpha|/S_0\gg S_0z/2$.

The result that transmission distortion approaches zero as the coupling S_0z approaches infinity is due to neglecting the frequency dependence of the coupling coefficient (C_0 of (14) is assumed independent of the frequency f of the signals on the transmission line). A companion paper [20] extends the present work to include the f dependence of C of (9); this extension is important in practical cases such as waveguides or fibers with white straightness deviation.

Appendix I

Frequency Dependence of $\Delta\beta$—Examples

Assume for purposes of illustration that the signal mode 0 has greater group velocity than the spurious mode 1 (the opposite case is readily treated). β_0, β_1, and $\Delta\beta=\beta_0-\beta_1$ are odd functions of frequency f. We consider two cases.

1) Dispersionless modes

$$\beta_0=\frac{2\pi f}{v_0}\qquad \beta_1=\frac{2\pi f}{v_1}\cdot \tag{147}$$

v_0 is both the group and phase velocity of the signal mode, and v_1 similarly for the spurious mode; both are constant (strictly independent of frequency). Since the signal mode is assumed faster,

$$v_0>v_1. \tag{148}$$

We have

$$\Delta\beta=-\left(\frac{1}{v_1}-\frac{1}{v_0}\right)2\pi f. \tag{149}$$

$\Delta\beta$ is strictly linear with f and has opposite sign and hence negative slope.

2) Waveguide-type modes: Throughout let the subscript x be 0 or 1, denoting signal or spurious mode, respectively. For positive frequencies, $f>0$:

$$\beta_x=\frac{2\pi f}{c}\sqrt{1-(f_x/f)^2} \tag{150}$$

$$v_{\phi x}=\frac{c}{\sqrt{1-(f_x/f)^2}}\qquad v_{gx}=c\sqrt{1-(f_x/f)^2}. \tag{151}$$

f_x are the cutoff frequencies, $v_{\phi x}$ the phase velocities, and v_{gx} the group velocities for the two modes; c is the free-space velocity. We have (for $f>0$):

$$f_0<f_1\qquad \beta_0>\beta_1\qquad v_{g0}>v_{g1}\qquad \Delta\beta>0$$

$$\frac{d}{df}\Delta\beta=2\pi\left(\frac{1}{v_{g0}}-\frac{1}{v_{g1}}\right)<0. \tag{152}$$

$\Delta\beta(f)$ again has negative slope [as in 1) above], but $\Delta\beta>0$ [unlike 1)]. Over a sufficiently narrow band $\Delta\beta$ varies approximately linearly with f (with negative slope); the farther from cutoff, the wider the band over which the linear approximation may be used.

Appendix II

Derivation of General Properties of Section IV

The general solution to (1)–(8) may be written as [8]

$$G_0(\Delta\Gamma)=1+\sum_{n=1}^{\infty}(-1)^n G_{0(n)}(\Delta\Gamma) \tag{153}$$

where

$$\begin{aligned}G_{0(n)}(\Delta\Gamma)&=\int_0^z dx_1\int_0^{x_1}dx_2\cdots\int_0^{x_{2n}}dx_{2n}\\&\quad\cdot c(x_1)c(x_2)\cdots c(x_{2n})\\&\quad\cdot\exp[\Delta\Gamma(x_1-x_2+x_3-x_4+\cdots-x_{2n})]\end{aligned} \tag{154}$$

where we take the coupling $c(x)$ as a fixed arbitrary function and the overall guide length z as a fixed parameter. The terms of the series are bounded by [8]

$$|G_{0(n)}(\Delta\Gamma)|\le\begin{cases}\dfrac{\left[\int_0^z|c(x)|\,dx\right]^{2n}}{(2n)!}, & \Delta\alpha\le 0.\\[2ex] \dfrac{\left[\int_0^z|c(x)|\,dx\right]^{2n}}{(2n)!}e^{\Delta\alpha z}, & \Delta\alpha\ge 0.\end{cases} \tag{155}$$

Under suitable conditions the $G_{0(n)}$ are analytic functions of $\Delta\Gamma$, the series for G_0 is uniformly convergent, and hence G_0 is an analytic function of $\Delta\Gamma$ [property 1)]. This result is a special case of a general theorem [14].[21]

[21] We are indebted to S. R. Neal for this reference.

Setting

$$s = x_1 - x_2 + x_3 - x_4 + \cdots - x_{2n} \tag{156}$$

in (154), we may write this relation as

$$G_{0(n)}(\Delta\alpha, \Delta\beta) = \int_0^z f_n(s) \exp(\Delta\alpha s) \exp(j\Delta\beta s) ds \tag{157}$$

where $f_n(s)$ is a $(2n-1)$-fold integral of the product of $2n$ c's, and therefore real. Normalizing the integration variable by setting

$$s = z\tau \tag{158}$$

(153) becomes

$$G_0(\Delta\alpha, \Delta\beta) = 1 + z \sum_{n=1}^{\infty} \int_0^1 f_n(z\tau) \exp(\Delta\alpha z\tau) \cdot \exp(j\Delta\beta z\tau) d\tau. \tag{159}$$

Interchanging the order of $\sum$ and $\int$, denoting $\sum_{n=1}^{\infty} f_n(z\tau) \equiv f(z\tau)$, and comparing with (11) and (12), the normalized impulse response is

$$g_{\Delta\alpha}(\tau) = \begin{cases} [\delta(\tau) + zf(z\tau)] \exp(\Delta\alpha z\tau), & 0 \le \tau \le 1 \\ 0, & \text{otherwise.} \end{cases} \tag{160}$$

This establishes properties 2)–5).

Alternatively, Laplace-transform relationships [15] may be used to establish these results from (153) and (154).

Appendix III
Derivation of (109)

We outline the calculations necessary to obtain the Fourier transform $P(\tau)$ (108) of $R_0(\sigma)$ of (70), which yields the normalized *pulse response* $P(-\tau)$ of the signal mode for the coupled line equations with white coupling.

Substituting in the result of [18]

$$f \to \frac{\sigma}{4\pi} \qquad a \to z \qquad g \to 2\tau z$$
$$\lambda \to \Delta\alpha + jS_0 \qquad \mu \to \Delta\alpha - jS_0 \tag{161}$$

we obtain the result

$$\int_{-\infty}^{\infty} \frac{\sinh(S_0 z\sqrt{1+\Sigma^2})}{\sqrt{1+\Sigma^2}} \exp(j\tau\sigma z) d\left(\frac{\sigma z}{2\pi}\right)$$
$$= S_0 z \cdot \exp(-2\Delta\alpha z\tau) I_0(S_0 z\sqrt{1-4\tau^2}), \qquad |\tau| < \tfrac{1}{2} \tag{162}$$

where, as in (70)

$$\Sigma \equiv \frac{\Delta\alpha + j\sigma/2}{S_0} \tag{163}$$

and I_0 represents a zeroth-order modified Bessel function. In (162) and in all subsequent relations of this appendix, all functions are $\equiv 0$ outside their range of definition (e.g., the left-hand side of (162) $\equiv 0$ for $|\tau| > \frac{1}{2}$).

Taking $[1/(2S_0 z)]\ d/d\tau$ of both sides of (162)

$$\int_{-\infty}^{\infty} \frac{j\sigma}{2S_0} \frac{\sinh(S_0 z\sqrt{1+\Sigma^2})}{\sqrt{1+\Sigma^2}} \exp(j\tau\sigma z) d\left(\frac{\sigma z}{2\pi}\right)$$
$$= \tfrac{1}{2}\exp(\Delta\alpha z)\cdot\delta(\tau+\tfrac{1}{2}) - \tfrac{1}{2}\exp(-\Delta\alpha z)\cdot\delta(\tau-\tfrac{1}{2})$$
$$- \Delta\alpha z \cdot \exp(-2\Delta\alpha z\tau) I_0(S_0 z\sqrt{1-4\tau^2})$$
$$- 2S_0 z \cdot \exp(-2\Delta\alpha z\tau) \frac{\tau}{\sqrt{1-4\tau^2}} I_1(S_0 z\sqrt{1-4\tau^2}), \qquad |\tau| \le \tfrac{1}{2} \tag{164}$$

I_0 and I_1 being zeroth- and first-order modified Bessel functions.

Next, using (163),

$$\int_{-\infty}^{\infty} \Sigma \frac{\sinh(S_0 z\sqrt{1+\Sigma^2})}{\sqrt{1+\Sigma^2}} \exp(j\tau\sigma z) d\left(\frac{\sigma z}{2\pi}\right)$$
$$= \frac{\Delta\alpha}{S_0} \times \text{eq. (162)} + \text{eq. (164)}$$
$$= \tfrac{1}{2}\exp(\Delta\alpha z)\cdot\delta(\tau+\tfrac{1}{2}) - \tfrac{1}{2}\exp(-\Delta\alpha z)\cdot\delta(\tau-\tfrac{1}{2})$$
$$- 2S_0 z \cdot \exp(-2\Delta\alpha z\tau) \frac{\tau}{\sqrt{1-4\tau^2}} I_1(S_0 z\sqrt{1-4\tau^2}), \qquad |\tau| \le \tfrac{1}{2}. \tag{165}$$

Further,

$$\frac{d}{d\sigma} \cosh(S_0 z\sqrt{1+\Sigma^2}) = \frac{jz}{2} \Sigma \frac{\sinh(S_0 z\sqrt{1+\Sigma^2})}{\sqrt{1+\Sigma^2}}. \tag{166}$$

Therefore,

$$\int_{-\infty}^{\infty} \cosh(S_0 z\sqrt{1+\Sigma^2}) \exp(j\tau\sigma z) d\left(\frac{\sigma z}{2\pi}\right) = -\frac{1}{2\tau} \times \text{eq. (165)}. \tag{167}$$

Combining (165) and (167), the transform of the [ ] factor of $R_0(\sigma)$ of (70) is

$$\int_{-\infty}^{\infty} \left[\cosh(S_0 z\sqrt{1+\Sigma^2}) - \Sigma \frac{\sinh(S_0 z\sqrt{1+\Sigma^2})}{\sqrt{1+\Sigma^2}}\right] \cdot \exp(j\tau\sigma z) d\left(\frac{\sigma z}{2\pi}\right)$$
$$= -\frac{1+2\tau}{2\tau} \times \text{eq. (165)}$$
$$= \exp(-\Delta\alpha z)\cdot\delta(\tau-\tfrac{1}{2})$$
$$+ S_0 z \cdot \exp(-2\Delta\alpha z\tau) \frac{1+2\tau}{\sqrt{1-4\tau^2}} I_1(S_0 z\sqrt{1-4\tau^2}), \qquad |\tau| \le \tfrac{1}{2}. \tag{168}$$

Finally, the transform of $R_0(\sigma)$ of (70) is

$$\int_{-\infty}^{\infty} R_0(\sigma) \exp (j\tau\sigma z) d\left(\frac{\sigma z}{2\pi}\right)$$

$$= \exp(-S_0 z) \cdot \exp (\Delta\alpha z) \times [\text{eq. (168) with } \tau \to \tau + \tfrac{1}{2}] \quad (169)$$

yielding (108) and (109).

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Optimum Coupling for Random Guides with Frequency-Dependent Coupling

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***Abstract*—We obtain exactly the covariance of the signal–signal and signal–spurious mode transfer functions of the coupled line equations with two forward-traveling modes, white random coupling with statistically independent successive values (e.g., white Gaussian or Poisson coupling), and a coupling coefficient that varies with the frequency of the signals on the line. No perturbation or other approximations are made in this work. Time-domain statistics for the corresponding impulse responses are obtained for moderate fractional bandwidths.**

These results are extensions of a similar treatment for frequency-independent coupling coefficients, given in a companion paper. If the coupling were independent of frequency, the signal distortion would ultimately decrease as the coupling increased, approaching zero as the coupling approached infinity. The frequency dependence of the coupling coefficient prevents the distortion from approaching zero; the optimum coupling, which achieves minimum signal distortion, is independent of guide length.

Millimeter waveguides and optical fibers with random straightness deviations have coupling coefficients inversely proportional to the frequency. The above results yield the optimum random straightness deviation for such a guide.

More forward modes can be treated in a straightforward way by more complicated calculations.

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I. Introduction

We study exactly the coupled line equations for signal and spurious modes (0 and 1) traveling in the forward direction [1]:

$$I_0'(z) = -\Gamma_0 I_0(z) + jc(z) I_1(z)$$
$$I_1'(z) = jc(z) I_0(z) - \Gamma_1 I_1(z) \quad (1)$$

subject to the initial conditions

$$I_0(0) = 1 \qquad I_1(0) = 0 \quad (2)$$

with coupling coefficient $c(z)$ proportional to a random