The Synthesis of Symmetrical Waveguide Circulators*

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Summary—A method for synthesizing symmetrical waveguide circulators by adjusting the eigenvalues of the scattering matrix is described. This procedure is particularly useful for the design of very compact circulators in the form of waveguide junctions containing ferrite obstacles. Permissible structural symmetries for a circulator are listed, and a standard form for the scattering matrix of a symmetrical circulator is defined. The synthesis procedure is then described in detail, stating the conditions to be imposed on the scattering matrix eigenvalues, and an expression is obtained for the changes in the eigenvalues due to the placing of anisotropic material within the junction.

By applying the theory to Allen's 4-port turnstile circulator, it is shown that the use of a matched turnstile junction and a reflectionless Faraday rotator is not essential. The theory is also applied to the design of novel 3- and 4-port circulators, and two 6-port circulators, one of which may be used as a 5-position waveguide switch, are described. Some experimental results are presented for a compact 3-port circulator in the form of an *H*-plane *Y* junction, in 1 inch by ½ inch waveguide, containing a ferrite post obstacle. This circulator, which operates with a bias field of approximately 25 oersted, has a useful bandwidth of 3 per cent. Greater bandwidths would be expected in a Stripline or a fin-line version of this device.

INTRODUCTION

THE usual approach to the synthesis problem for microwave circuits is to use lumped-element network representations of the waveguide components. A lumped-element network having the desired characteristics is first obtained by conventional synthesis procedures, and a microwave structure exhibiting the same characteristics is then synthesized by interconnecting basic waveguide components, represented by lumped element networks and sections of transmission line. This method has been extended by Carlin[1] to the synthesis at a single frequency of non-reciprocal networks. At microwave frequencies the physical structure of the circuit is not defined by the network diagram as at lower frequencies, and for this reason there is often no particular advantage in working from an explicit network representation of the desired circuit.

A more direct approach is to consider the microwave circuit as a waveguide junction characterized by a scattering matrix S, without reference to a specific network representation. The scattering matrix coefficients are then fixed in accordance with the desired circuit characteristics by suitably adjusting the geometry of the boundary-value problem. It has been shown by Dicke[2] that this is a useful approach when the microwave circuit has structural symmetry, in which case it is convenient to regard the eigenvectors and eigenvalues of S, rather than the scattering coefficients themselves, as adjustable parameters of the circuit. Given a specific scattering matrix S_d to be realized by a symmetrical junction at a single frequency, the synthesis procedure is as follows:

a) Determine which structural symmetries are consistent with the form of the matrix S_d.
b) Find the eigenvectors and eigenvalues of S_d.
c) Select a junction having one of the symmetries found in a) and, by making symmetrical perturbations of the junction geometry, adjust the scattering matrix eigenvectors and eigenvalues so as to coincide with b).

As in the case of the equivalent network approach, the procedure outlined above does not lead to a specific microwave circuit. However, once one of the permissible symmetries in a) has been chosen, the *nature* of the adjustments required in c) is strongly suggested by symmetry. The *range* of adjustment required must be determined experimentally. This procedure is particularly useful in the design of very compact structures which perform rather complicated circuit functions, such as the magic-T hybrid and the waveguide Wheatstone bridge.[3] However, it is also applicable to the design of symmetrical circuits which comprise interconnections of distinct waveguide components. The procedure provides for the realization of the desired circuit characteristics at a single frequency only. Nevertheless, as is usual with microwave circuits designed at a single frequency, the circuit will function with somewhat degraded performance over a band of frequencies in the vicinity of the design frequency. The bandwidth depends on the minimum performance specifications and also on details of the circuit structure not under direct control of the synthesis procedure. For example, a Stripline or a fin-line structure would be expected to have greater bandwidth than a rectangular waveguide structure with the same symmetry. A disadvantage of the symmetry method of synthesis is that circuit losses are not easily taken into account. Consequently, it is mainly applicable to circuits in which the losses are small and incidental.

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[1] H. J. Carlin, "Principles of Gyrator Networks," *Proc. Symp. on Modern Advances in Microwave Techniques*, Brooklyn Inst. Polytech., pp. 175–204; November, 1954.

[2] C. Montgomery, R. H. Dicke, and E. M. Purcell, "Principles of Microwave Circuits," McGraw-Hill Book Co., Inc., New York, N. Y., ch. 12; 1948.

[3] M. Chodorow, E. L. Ginzton, and J. F. Kane, "A microwave impedance bridge," PROC. IRE, vol. 37, pp. 634–639; June, 1949.

Although symmetrical circulators of several types have been described in the literature by Allen,[4] Fowler,[5] Vartanian,[6] and Kock,[7] symmetry methods have not been widely applied to non-reciprocal networks. However, Treuhaft[8] has described some of the general symmetry properties of circulators, and Fowler's circulator was designed by adjusting the eigenvalues of S, although full use was not made of the advantages of the method. The purpose of this paper is to apply the synthesis procedure outlined above to symmetrical waveguide circulators in general. In this connection the term "waveguide" is understood to include TEM structures such as coaxial line and Stripline.

Theory of Symmetrical Circulators

Permissible Symmetries

It has been shown by Dicke[2] that the scattering matrix S of any symmetrical junction must satisfy a set of commutation relations,

$$F_{g_i}S = SF_{g_i}, \qquad (1)$$

which determine the restrictions imposed on S by the junction symmetry. In (1) the F_{g_i}'s are a set of independent generators for a group of "symmetry operators" F for the junction. The "symmetry operators" F are matrices which indicate how the terminal fields transform under the operations of the symmetry group of the junction, *i.e.*, rotations and reflections that carry the junction into itself. For a detailed discussion of the properties of the F-matrices the above-mentioned reference and others[9,10] may be consulted. In the present application it is sufficient to note that the group of F-matrices constitute a "representation" of the symmetry group of the junction. That is, if F_1 is the matrix corresponding to a symmetry operation T_1, and F_2 corresponds to T_2, then the matrix product F_1F_2 corresponds to the symmetry operation T_1T_2 (operation T_2 followed by operation T_1). For an n-port circulator, defined as a device whose scattering matrix S_{cn} performs a cyclic substitution on the n incident wave amplitudes, the group of F-matrices can include only 1) matrices contained in any one of the subgroups of the cyclic group generated by S_{cn}; and 2) the products of these matrices with $-I$.[8] Since a cyclic group possesses only cyclic subgroups,[8] the only possible F-groups for a circulator are those generated by the following sets of generators:

a) $F_1 = (S_{cn})^p$ (p a factor of n), and $F_2 = -I$

$$F_1^{n/p} = F_2^2 = I, \qquad F_1F_2 = F_2F_1,$$

b) $F_1 = (S_{cn})^p$ (p a factor of n)

$$F_1^{n/p} = I,$$

c) $F_2 = -I$

$$F_2^2 = I,$$

where case c) is trivial, since every $n \times n$ matrix commutes with $-I$. It is found by consulting tabulations of all the possible symmetry groups for a physical structure (3-dimensional point groups),[10,11] that only those symmetry groups listed in Table I, where the usual Schoenflies notation is used, can be represented by the groups of F-matrices a) and b). Accordingly, these are the only permissible non-trivial symmetries for an n-port circulator (Fig. 1). It should be noted that the boundaries of the junction may possess more than the permitted number of symmetries, these extra symmetries being removed by the anisotropic material placed within the junction in order to induce nonreciprocity. In Figs. 1(a), (b), (c), and (d) for example, the vertical planes of symmetry for the junction boundaries might be removed by a circular ferrite post placed along the axis of symmetry of the junction, with a bias field parallel to the symmetry axis. It is demonstrated in the theory of finite groups[12] that when a cyclic substitution on n symbols is raised to a power $p = n/q$ the result is a noncyclic substitution consisting of p cycles, each operating on a distinct set of q symbols. Consequently, when $p \neq 1$ the "symmetry operator" F_1 permutes the junction ports *within* sets of $q = n/p$ ports but does not interchange sets of ports, as in Figs. 1(d), (e), and (f).

TABLE I

PERMISSIBLE SYMMETRIES FOR AN n-PORT CIRCULATOR

	(p is a factor of n)
(a)	$C_{(n/p)h}$
(b)	$C_{n/p}$, $S_{n/p}$ (n/p even), $C_{(n/2p)h}$ ($n/2p$ odd), C_i ($p = n/2$)

Treuhaft's definition of a circulator does not specify uniquely the form of the scattering matrix, and for a given n-port circulator different systems of port numbering will lead to $(n-1)!$ distinct scattering matrices. In the case of a structurally symmetrical circulator the number of possibilities may be reduced, however, by specifying a standard numbering system. For an n-port

[4] P. J. Allen, "The turnstile circulator," IRE TRANS. ON MICROWAVE THEORY AND TECHNIQUES, vol. MTT-4, pp. 223–228; October, 1956.

[5] H. Fowler, paper presented at Symposium on Microwave Properties and Applications of Ferrites, Harvard Univ., Cambridge, Mass.; April, 1956.

[6] P. H. Vartanian, "Theory and Applications of Ferrites at Microwave Frequencies," Sylvania Electronic Defense Lab. Rep. E 15, pp. 119–126; April, 1956.

[7] W. E. Kock, "Signal Routing Apparatus," U. S. Patent No. 2,794,172; May 28, 1957.

[8] M. A. Treuhaft, "Network properties of circulators based on the scattering concept," IRE TRANS. ON CIRCUIT THEORY, vol. CT-3, pp. 127–135; June, 1956. See also PROC. IRE, vol. 44, pp. 1394–1402; October, 1956.

[9] D. M. Kerns, "Analysis of symmetrical waveguide junctions," *J. of Research Nat. Bur. of Standards*, vol. 46, pp. 267–282; April, 1951.

[10] B. A. Auld, "Applications of Group Theory in the Study of Symmetrical Waveguide Junctions," MLR-157, Stanford University, Stanford, Calif.; March, 1952.

[11] F. D. Murnaghan, "The Theory of Group Representations," The Johns Hopkins Press, Baltimore, Md.; 1938.

[12] H. Hilton, "An Introduction to the Theory of Groups of Finite Order," Clarendon Press, Oxford, Eng., p. 8; 1908.

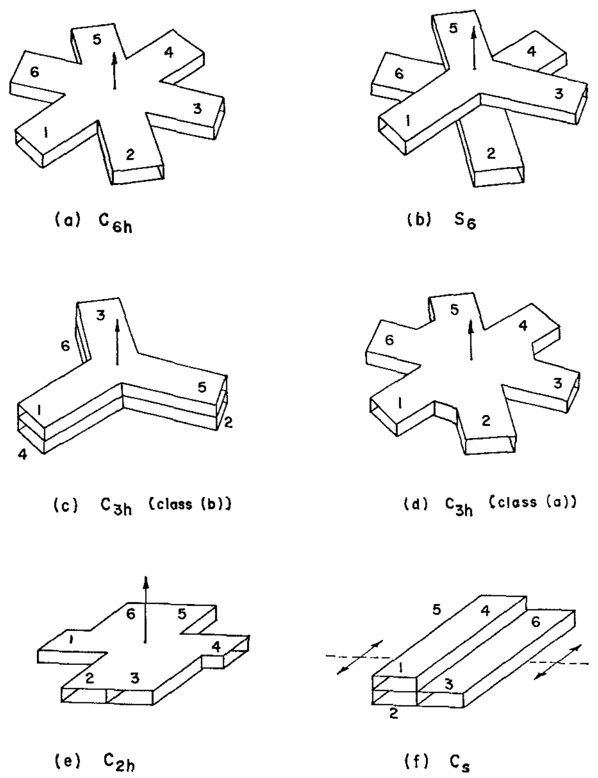


Fig. 1—Examples of some permissible symmetries for a 6-port circulator.

junction having any of the symmetries listed in Table I it is always possible to number the port, so that the generators of the F-group are $F_1=K_n^p$ and $F_2=-I$, where K_n is the matrix representing the cyclic substitution $(1, 2, 3, \cdots, n)$, and K_n^p represents, therefore, the substitution[12]

$$(1, 1+p, 1+2p, \cdots, 1+[q-1]p) \cdots \cdots (r, r+p, r+2p, \cdots, r+[q-1]p) \cdots \cdots (p, 2p, 3p, \cdots, n).$$

This is, in fact, a natural system of port numbering for such junctions (Fig. 1). If the ports are numbered according to this system, the form of the scattering matrix S of the junction is determined by the commutation relation[13]

$$SK_n^p = K_n^p S, \qquad (2)$$

and the scattering matrices S_{scn} of all symmetrical circulators are then restricted to those cyclic substitution matrices which satisfy (2). This may be taken as the standard form for the scattering matrix of a symmetrical circulator. When $p=1$, which will be shown to be the most useful case in practice, S_{scn} commutes with the *cyclic* substitution matrix K_n, and the standard form may be given explicitly. That is,

$$S_{scn} = K_n^f (f = 1, \text{ or is prime to } n), \qquad (3)$$

[13] The second commutation relation, $S(-I)=(-I)S$, is redundant, since it is satisfied by any matrix S.

since the only substitutions on n letters commutative with K_n are K_n^f, and K_n^f is cyclic only if $f=1$ or is prime to n.[8,12]

The Synthesis Procedure

The procedure used for the synthesis of symmetrical circulators follows Dicke's work on symmetrical *reciprocal* junctions. In essence, the method requires finding the eigenvectors and eigenvalues of the desired scattering matrix, and then making symmetrical perturbations of the junction geometry until the eigenvectors and eigenvalues of the junction scattering matrix coincide with those of the desired scattering matrix. Each junction eigenvector v and its associated eigenvalue s specify a set of incident wave amplitudes v and reflected wave amplitudes sv at the terminal planes of the junction, and therefore describe uniquely a solution (eigensolution) to the junction boundary-value problem. For a lossless junction $|s|=1$,[2] and each eigensolution is characterized by pure standing waves in the arms of the junction, the amplitudes of the fields in the arms being specified by the elements of the eigenvector, and the reflection coefficient at the terminal planes by the associated eigenvalue. In determining the *nature* of the perturbations required for adjusting the eigensolutions, it is necessary to make use of the symmetry properties of the eigensolutions. For this reason the synthesis procedure is of greatest practical use when the eigenvectors are completely determined by the symmetry of the junction, and the discussion will be limited to this case.

Let S be the scattering matrix of any lossless n-port junction having one of the permissible symmetries listed in Table I. If the ports are numbered appropriately, S satisfies the particular commutation relation (2), from which the symmetry restrictions on the eigenvectors and eigenvalues of S may be derived by means of the theory of group representations.[9,10] It is found that the eigenvectors are completely determined by symmetry only for symmetry groups C_{nh}, C_n, S_n (n even), or $C_{(n/2)h}$ ($n/2$ odd), that is, when $p=1$ in Table I. In this case, the general form of the scattering matrix eigenvectors is given by the set of column matrices u_a:

$$(u_a)_i = \frac{e^{-j2\pi a(i-1)/n}}{\sqrt{n}} \qquad (4)$$

$$a = 0, \pm 1, \pm 2, \cdots, \left(\frac{n}{2}\right)$$

$$i = 1, 2, 3, \cdots, n,$$

and the associated eigenvalues s_a are, in general, nondegenerate. Since $p=1$, the standard form for the scattering matrix of a circulator with one of the above symmetries is K_n^f, and the first step in the synthesis procedure is to find the eigenvectors and eigenvalues of this matrix.

For the matrix K_n the characteristic equation is

$$|K_n - \lambda I| = |\lambda^n - 1| = 0$$

from which the eigenvalues are

$$\lambda_a = \epsilon^{-j2\pi a/n}$$
$$a = 0, \pm 1, \pm 2, \cdots, \left(\frac{n}{2}\right). \tag{5}$$

The eigenvectors v_a are found by substituting (5) into

$$K_n v_a = \lambda_a v_a$$

from which it follows directly that

$$(v_a)_i = \epsilon^{-j2\pi a/n}(v_a)_{i-1}.$$

Taking $(v_a)_1 = 1/\sqrt{n}$, a set of normalized eigenvectors,

$$(v_a)_i = \frac{\epsilon^{-j2\pi a(i-1)/n}}{\sqrt{n}} \tag{6}$$

is obtained. These are also eigenvectors for the matrix K_n^f, with associated eigenvalues

$$\lambda_a^f = \epsilon^{-j2\pi af/n}. \tag{7}$$

Since the junction eigenvectors (4) already coincide with the required circulator eigenvectors (6), the final step in the synthesis procedure is to adjust the junction eigenvalues s_a in accordance with (7). It should be noted that the s_a's may all be rotated through an arbitrary angle θ by moving all the terminal reference planes through the electrical angle $\theta/2$. Consequently, only the relative phase angles of the eigenvalues are of importance, and the circulator conditions may be written

$$s_0 = \epsilon^{-j2\pi af/n} \qquad \left(a = \pm 1, \pm 2, \cdots, \left(\frac{n}{2}\right)\right), \tag{8}$$

where $f = 1$ or is prime to n. When the eigenvalues have been adjusted so as to satisfy (8), the scattering matrix of the junction takes the desired form

$$S_{scn} = \epsilon^{j\theta} K_n^f$$

where the exponential factor allows for an arbitrary shift of the terminal reference planes.

Perturbation Theorem for the Scattering Matrix Eigenvalues

It has been shown that to realize a circulator in the form of a symmetrical junction it is necessary to perturb the junction geometry so as to fulfill condition (8). The perturbations take the form of deformations of the walls and the placing of isotropic and anistropic materials within the junction, due regard being paid to the preservation of symmetry in order that the eigenvectors (4) are not disturbed. Changes in the eigenvalues due to a particular perturbation are functions of both the nature of the perturbation and its position within the junction, and may be expressed as integrals of the eigensolution fields over the volume of the perturbation.

Assuming that the parameters within a volume V_1 inside the junction are changed from μ, ϵ (scalars) to $\overleftrightarrow{\mu}'$, $\overleftrightarrow{\epsilon}'$ (either scalars or tensors), let $E^{(-a)}$, $H^{(-a)}$ represent the electromagnetic field of the $-a$th eigensolution for the unperturbed junction, and let $E^{(a)\prime}$ $H^{(a)\prime}$ represent the electromagnetic field of the ath eigensolution for the perturbed junction. Then

$$\nabla \cdot (E^{(-a)} \times H^{(a)\prime} - E^{(a)\prime} \times H^{(-a)}) = j\omega(H^{(-a)} \cdot \overleftrightarrow{\delta\mu} H^{(a)\prime} - E^{(-a)} \cdot \overleftrightarrow{\delta\epsilon} E^{(a)\prime}) \tag{9}$$

where $\overleftrightarrow{\delta\mu} = \overleftrightarrow{\mu}' - \mu$ and $\overleftrightarrow{\delta\epsilon} = \overleftrightarrow{\epsilon}' - \epsilon$. Eq. (9) is integrated over the volume bounded by the junction and the left-hand side is converted in the usual manner into a sum of surface integrals over the terminal planes. Since the fields are eigensolutions, this sum may be expressed in matrix notation as

$$\sum_i \int_{S_i} (E_i^{(-a)} \times H_i^{(a)\prime} - E_i^{(a)\prime} \times H_i^{(-a)}) \cdot n_i dS_i = 2(s_a' - s_{-a}) \tilde{u}_{-a} u_a$$

where the symbol $\tilde{u}_{-a}$ means the transpose of the column matrix u_{-a}. From (4)

$$\tilde{u}_{-a} u_a = 1.$$

Hence,

$$s_a' - s_{-a} = j\frac{\omega}{2}\int_{V_1} (H^{(-a)} \cdot \overleftrightarrow{\delta\mu} H^{(a)\prime} - E^{(-a)} \cdot \overleftrightarrow{\delta\epsilon} E^{(a)\prime}) dV. \tag{10}$$

For the ath and $-a$th eigensolutions of the unperturbed junction,

$$S u_a = s_a u_a$$
$$S u_{-a} = s_{-a} u_{-a}.$$

Multiplying the transpose of the second equation by u_a on the right and the first equation by $\tilde{u}_{-a}$ on the left, and subtracting,

$$\tilde{u}_{-a}(\tilde{S} - S) u_a = (s_{-a} - s_a)$$

Since the unperturbed junction is assumed to contain only isotropic material, $\tilde{S} = S$ and it follows that $s_a = s_{-a}$ (reciprocity degeneracy).[9] Substituting this result into (10),

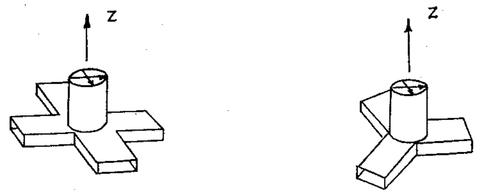
$$\delta s_a = s_a' - s_a = j\frac{\omega}{2}\int_{V_1} (H^{(-a)} \cdot \overleftrightarrow{\delta\mu} H^{(a)\prime} - E^{(-a)} \cdot \overleftrightarrow{\delta\epsilon} E^{(a)\prime}) dV \tag{11}$$

which gives the change of the ath eigenvalue due to the perturbation.

Applications

Turnstile Circulators

A familiar example of a circulator with symmetry group C_4 is Allen's[4] 4-port circulator, based on the well-known 4-arm turnstile junction [Fig. 2(a)]. In this circulator a *matched* turnstile junction is used, and the circular arm is terminated by a Faraday rotator in front of an adjustable short-circuit plunger. Allen treated the device as an interconnection of two distinct



(a) (b)

Fig. 2—(a) Four-arm turnstile junction. (b) Three-arm turnstile junction.

components and, assuming that the Faraday rotator was reflectionless and did not extend into the fringing fields of the turnstile, demonstrated that circulator action can be achieved by suitably adjusting the plane of polarization and the phase of the reflected wave in the circular arm. An alternative approach would be to apply the symmetry theory outlined above, the scattering matrix eigenvalues being adjusted in accordance with (8) by means of the Faraday rotator and the sliding short-circuit. This approach, which might appear to be unnecessarily complicated, leads to several conclusions not readily apparent from Allen's analysis.

When applying symmetry theory to a composite junction, it is often convenient to express the composite eigensolutions as linear combinations of eigensolutions for the component parts of the junction, thereby arriving at expressions for the eigenvalues of the composite junction in terms of the eigenvalues of its component parts. In the case of the turnstile circulator, no initial assumption is made regarding the matching of the turnstile junction. The composite eigensolutions are constructed by taking linear combinations of the turnstile eigensolutions, subject to restrictions imposed by the load attached to the circular arm. Relations between the composite eigenvalues s_a and the turnstile eigenvalues s_{bt} are obtained, and certain circulator conditions on the s_{bt}'s are derived by applying the circulator conditions (8) to the s_a's.

Several general conclusions may be drawn from such an analysis.[14] In the first place, the ferrite element does not have to be matched. If the circular pipe propagates only the lowest mode and the fringing fields of the termination and the turnstile do not interact, the termination does not even have to be of the same symmetry as the turnstile. The ferrite element may, in fact, have any n-fold rotational symmetry ($n \neq 2$). Furthermore, matching of the turnstile is not essential. It is necessary only to insert a thin metal pin along the axis of symmetry, and this adjustment is not sufficient to match the turnstile.[15] By deliberately mismatching the turnstile, thereby converting the terminated circular pipe into a resonant cavity, a reduction in the volume of ferrite required may be obtained at the expense of bandwidth. Similar results are obtained in the case of Fowler's circulator, based on the 3-arm turnstile junction [Fig. 2(b)].

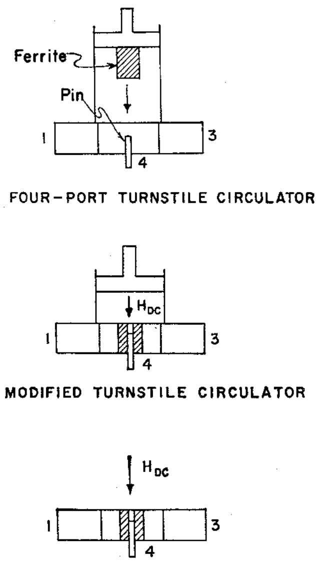


Fig. 3—Modifications of the 4-port turnstile junction.

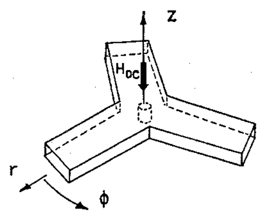


Fig. 4—Three-port star circulator.

The above discussion has been based on the explicit assumption that the fringing fields of the ferrite element and the turnstile do not interact. This restriction is not essential, and circulator action can still be realized even when the ferrite element is placed entirely within the junction region [Fig. 3(b)]. The advantage of this is that the short-circuit plunger can now be placed closer to the junction, shortening the electrical length of the side arm and reducing the frequency sensitivity of the circulator.[16] Since the short circuit plunger is placed outside the fringing fields, the composite junction theory may again be used. However, in this case the results are more complicated, owing to the nonreciprocity of the turnstile junction itself; and the adjustment procedure is consequently more difficult.

Star Junction Circulators

Further reduction of frequency sensitivity is achieved if the plunger and the ferrite element are both placed within the fringing fields, resulting in a star junction structure [Fig. 3(c)]. This structure cannot be treated as a

[14] B. A. Auld, "The Synthesis of Symmetrical Waveguide Circulators," Electrical Engineering Report, Univ. of British Columbia, Vancouver, Canada; Aug., 1958.

[15] Montgomery, Dicke, and Purcell, *op. cit.*, pp. 459–466.

[16] The writer has recently learned that another method for increasing the bandwidth of a turnstile circulator has been used very successfully in the 3-port case by T. Schaug-Pettersen ("Novel Design of a 3-Port Circulator," Internal Rep., Norwegian Defence Res. Est., Bergen, Norway; January, 1958). In this method the phase-frequency response curves for all the scattering matrix eigenvalues are adjusted so as to approximately coincide over a band of frequencies, thus permitting the circulator conditions (8) to be approximately satisfied over this entire frequency band.

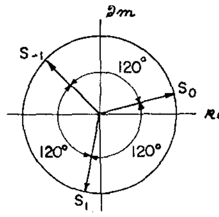


Fig. 5—Scattering matrix eigenvalues for a 3-port circulator.

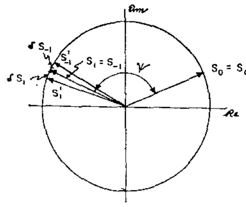


Fig. 6—Splitting of the reciprocity degeneracy due to the application of a small bias field.

composite junction, and the circulator conditions (8) are established by altering the geometry of the junction region. As an example of this type of circulator, a 3-port circulator in the form of a symmetrical *H*-plane *Y*-junction with a ferrite post across the guide on the axis of symmetry (Fig. 4), will be considered.

The circulator conditions for a 3-port junction are, from (8),

$$s_1 = \epsilon^{\mp j2\pi/3} s_0$$
$$s_{-1} = \epsilon^{\pm j2\pi/3} s_0,$$

the upper signs corresponding to circulation in the direction 1→2→3 (Fig. 5), and the lower signs to circulation in the reverse direction. To satisfy these conditions, it is necessary to set only two of the junction parameters, the phases of s_1 and s_{-1} relative to s_0, and only two physical adjustments are required. These may be taken to be the diameter of the ferrite post and the strength of the bias field.

Let it be assumed that with no bias field applied, the eigenvalues are s_0 and $s_1 = s_{-1}$ (Fig. 6). Assuming no losses, these will lie on the unit circle in the complex plane with a relative angular spacing ψ, which is a function of the post diameter. The eigenvalues are, in effect, reflection coefficients and must have phase angles which increase with frequency. Consequently, s_0 and $s_1 = s_{-1}$ rotate around the unit circle in a counter-clockwise direction with increasing frequency, and, since the phase angles of s_0 and $s_1 = s_{-1}$ will not generally have the same frequency dependence, the angular spacing ψ will vary with frequency as well as with post diameter.

In general, the effect of applying bias field [$\overleftrightarrow{\delta\epsilon} = 0$ in (11)] is to alter the phases of all the eigenvalues. Because of this, the circulator conditions cannot be set up by a straightforward balancing procedure as in the case of the turnstile circulator. However, if only small perturbations are considered and second-order effects neglected, the results are sufficiently simple to serve as the basis for a fairly systematic adjustment procedure. When second-order terms are neglected, (11) becomes

$$\delta s_a = j \frac{\omega}{2} \int_{V_1} \boldsymbol{H}^{(-a)} \cdot \overleftrightarrow{\delta\mu} \boldsymbol{H}^{(a)} dV \qquad (a = 0, +1, -1). \quad (12)$$

If the internal bias field is assumed to be uniform, the biased ferrite medium is characterized by the Polder permeability tensor, and

$$\delta\mu = \begin{bmatrix} \mu - \mu_u & j\kappa & 0 \\ -j\kappa & \mu - \mu_u & 0 \\ 0 & 0 & \mu_z - \mu_u \end{bmatrix} \quad (13)$$

where μ_u is the scalar permeability of the unbiased ferrite. The terminal conditions for the eigensolution fields $\boldsymbol{E}^{(a)}$, $\boldsymbol{H}^{(a)}$ are defined by the scattering matrix eigenvectors, in this case

$$u_0 = \frac{1}{\sqrt{3}} \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix} \qquad u_1 = \frac{1}{\sqrt{3}} \begin{bmatrix} 1 \\ \epsilon^{-j2\pi/3} \\ \epsilon^{j2\pi/3} \end{bmatrix}$$

$$u_{-1} = \frac{1}{\sqrt{3}} \begin{bmatrix} 1 \\ \epsilon^{j2\pi/3} \\ \epsilon^{-j2\pi/3} \end{bmatrix}. \quad (14)$$

Accordingly, the electric fields of the eigensolutions must satisfy relations

$$E_z^{(a)}(r, \phi) = \epsilon^{j2\pi am/3} \quad E_z^{(a)}\left(r, \phi + \frac{2\pi m}{3}\right)$$
$$(a = 0, +1, -1), \qquad (m \text{ any integer})$$

where only the z component appears because the structure is uniform in the z direction. The eigensolution electric fields within the ferrite post must therefore take the form

$$E_z^{(a)} = \sum_{p=-\infty}^{\infty} A_p^{(a)} J_{|a+3p|}(kr) \epsilon^{-j(a+3p)\phi}$$

where $k = \omega\sqrt{\mu_u \epsilon}$ and the magnetic fields are of the form

$$\boldsymbol{H}^{(a)} = -\frac{\nabla \times k E_z^{(a)}}{j\omega\mu_u}$$
$$= \frac{-\epsilon^{-ja\phi}}{j\omega\mu_u} \sum_{p=-\infty}^{\infty} A_p^{(a)} \boldsymbol{f}_p^{(a)}(kr) \epsilon^{-j3p\phi}. \quad (15)$$

Substituting (13) and (15) into (12),

$$\delta s_a = -j \frac{b}{2\omega\mu_u^2} \sum_{p=-\infty}^{\infty} A_{-p}^{(-a)} A_p^{(a)}$$
$$\cdot \left\{ (\mu - \mu_u) \int_0^R \boldsymbol{f}_{-p}^{(-a)} \cdot \boldsymbol{f}_p^{(a)} r dr \right.$$
$$\left. + j\kappa \int_0^R \boldsymbol{k} \cdot \boldsymbol{f}_{-p}^{(-a)} \times \boldsymbol{f}_p^{(a)} r dr \right\} \qquad (a \neq 0)$$

$$\delta s_0 = -j \frac{b}{2\omega\mu_u^2} \sum_{p=-\infty}^{\infty} A_{-p}^{(0)} A_p^{(0)} (\mu - \mu_u)$$
$$\cdot \int_0^R \boldsymbol{f}_{-p}^{(0)} \cdot \boldsymbol{f}_p^{(0)} r dr \quad (16)$$

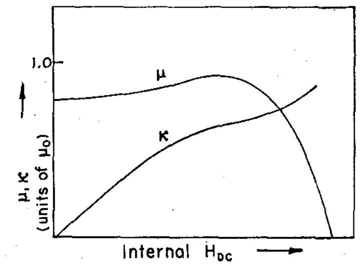
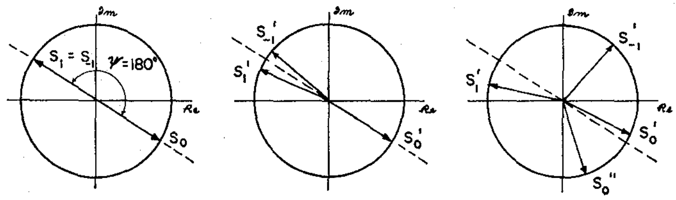


Fig. 7—Typical variation of ferrite permeability components as a function of internal bias field.

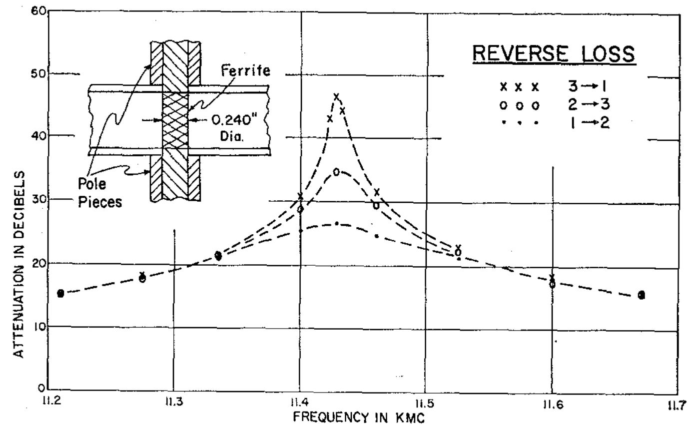


(a) (b) (c)

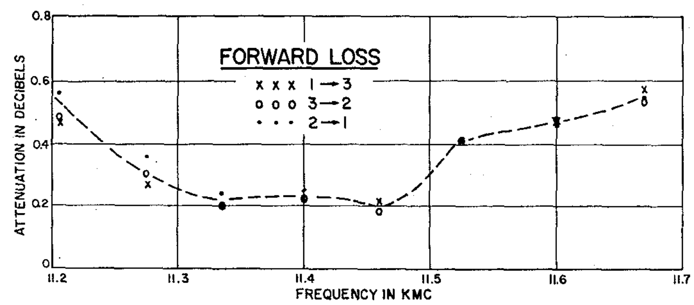
Fig. 8—Stages in the adjustment of a 3-port star circulator.

where R is the radius of the ferrite-post and b is the height of the waveguide.

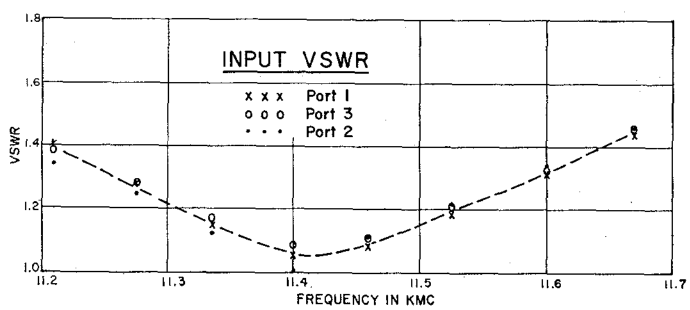
For a typical ferrite, the components μ, κ of the Polder tensor vary with applied field as shown in Fig. 7. It is seen from this that for small bias fields $\mu - \mu_u = 0$, and the changes in the eigenvalues are $\delta s_0 = 0$ and $\delta s_1 = -\delta s_{-1}$ (Fig. 6). In attempting to establish the conditions for circulation, it is desirable, before applying the bias field, to adjust the post diameter so that $\psi = 180°$ [Fig. 8(a)]. (This condition is satisfied by determining experimentally the post diameter d_{opt} required for optimum matching of the unbiased junction; from the Appendix, $|S_{11}|_{\text{opt}} = \frac{1}{3}$.) Then, when a small bias field is applied, the reciprocity degeneracy is split symmetrically [Fig. 8(b)], the amount of splitting increasing with the strength of the bias field. However, as the separation of s_1' and s_{-1}' is increased by increasing the bias field, terms in $(\mu - \mu_u)$ can no longer be neglected in (16), and asymmetrical splitting, as well as a change in s_0, occurs. Higher order perturbation terms in (11) will contribute further to this effect. Consequently when the required 120° splitting of s_1', s_{-1}' has been reached, s_0' will not lie in the position required for circulation [Fig. 8(c)]. To obtain circulator action, the angle ψ must be adjusted, by varying either the frequency or the post diameter, so as to compensate for this effect. If a post diameter d_{opt} is used, circulation may be established by varying the frequency and the bias field until the junction is matched. Alternatively, the circulator may be adjusted at a fixed frequency by selecting various post diameters in the neighborhood of d_{opt} and varying the bias field until a combination is found for which the junction is matched. The adjustment procedure is simplified greatly if a third adjustment, a thin metal pin along the axis of symmetry, is added. Since the pin affects only the 0th eigensolution it may be used to shift s_0' in Fig. 8(c) to its correct position s_0'', without changing s_1' and s_{-1}'; and circulation is obtained by alternately adjusting the pin and the bias field for minimum reflection until a match is obtained.



(a)



(b)



(c)

Fig. 9—Typical performance curves for a 3-port star circulator in 1 inch by $\frac{1}{2}$ inch waveguide. $H_{dc} \approx 25$ oersted in the empty gap.

In practice, ferrite circulator performance is degraded to some extent by incidental losses in the ferrite, which have not been taken into account in the above theory. Consequently, the synthesis procedure will be useful in practice only if the eigenvalues can be adjusted over a sufficiently large range without biasing the ferrite into the region of high losses. For the 8.2 to 12.4 kmc band the usefulness of the procedure has been demonstrated by the construction of 3-port star junction circulators in 1 by $\frac{1}{2}$ inch waveguide. In these circulators the ferrite element, a cylindrical post across the waveguide (Fig. 4), was placed directly between the magnet pole pieces

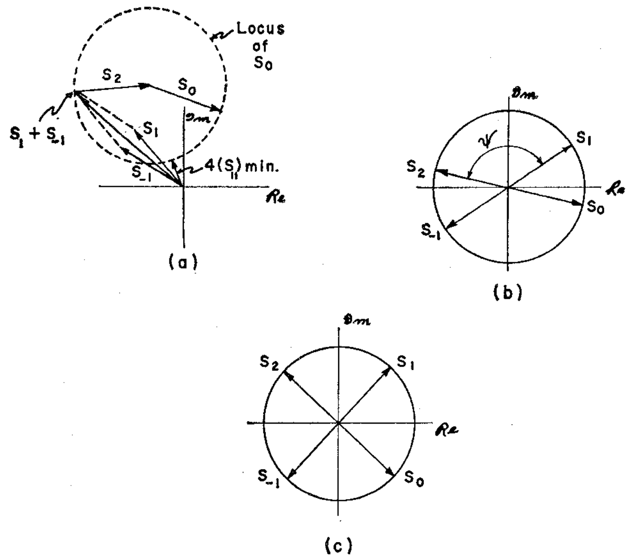


Fig. 10—Stages in the adjustment of a 4-port star circulator.

so as to form a closed magnetic circuit. With Ferramic R-1 posts the bias field strengths required were found to be less than 100 oersted *in the empty gap*. Typical performance curves are presented in Fig. 9, showing the effect of small asymmetries in the structure. In the case of one of the circulators tested it was found that the effects of asymmetries could be partially cancelled by rotating the ferrite post to an optimum position. The peaks of the three reverse loss curves were then all greater than 40 db, with no deterioration of the forward loss or VSWR characteristics.[17]

The synthesis of a 4-port star circulator would proceed in similar fashion. For a 4-port junction the circulator conditions are given by (8) with $n=4$, Fig. 10(c). To meet these conditions, it is necessary to set three of the junction parameters, and three physical adjustments are required. As in the case of the 3-port, two of these adjustments may be the post diameter and the bias field strength. A convenient choice for the third adjustment is the length of a thin metal pin on the axis of symmetry. This perturbs only the 0th eigensolution and permits the setting of s_0 independently. Since the other two adjustments change the phase angles of all four eigenvalues, the adjustment procedure is necessarily somewhat involved. With a given post diameter, bias field is applied, splitting the degeneracy $s_1=s_{-1}$; and $4|S_{11}|=|s_0+s_1+s_{-1}+s_2|$ (see Appendix) is minimized by adjusting the pin [Fig. 10a]. The splitting of s_1 and s_{-1} is increased in steps by increasing the bias field, the pin being adjusted for minimum $|S_{11}|$ at each step, until both $(S_{11})_{\min}$ and S_{13} vanish for the same pin setting. This can occur only when the splitting of s_1 and s_{-1} is 180°, so that $S_{11}=S_{13}=(s_0+s_2)/4$ (see Appendix), the null condition then being $s_2=-s_0$ [Fig. 10(b)]. Unless the initial choice of post diameter has been very fortunate, the angle ψ will not be 90°, and the procedure must then be repeated for various post diameters until

$$S_{12}=\frac{s_0+js_1}{2}=0, \text{ or } S_{21}=\frac{s_0-js_1}{2}=0.$$

At this point, $s_1=\pm js_0$ and circulation is established [Fig. 10(c)]. This result can be achieved only if the metal pin provides a sufficiently large range of adjustment of s_0. If this is not the case, other methods for adjusting s_0 must be considered.

The synthesis procedure may, in principle, be applied to circulators having any number of ports. In practice, however, the difficulties increase rapidly with the number of ports. Taking 5- and 6-port star circulators as examples, it is seen from the circulator conditions (8) that in both cases it is necessary to split two sets of reciprocity degenerate eigenvalues. This requires the use of two biased ferrite elements, and results in greatly increased complexity of the adjustment procedure. Fortunately, it may be shown that 6- and 8-port symmetrical circulators can easily be obtained without having recourse to the theory.

Symmetrical Ring Circulators

An alternative physical arrangement for an n-port circulator with symmetry group C_n is the symmetrical ring structure. Vartanian[6] has proposed a 3-port circulator of this type, using three gyrators, and a similar device is described by Kock.[7] Design formulas for the analogous 4-port symmetrical ring circulator, using four gyrators, are easily derived by regarding the circuit as a composite junction and applying symmetry theory, as in the case of the turnstile circulator. However, a more useful type of symmetrical ring circulator is obtained by simply interconnecting six 3-port star or turnstile circulators [Fig. 11(a)]. If the bias fields for the six circulators all have the same polarity, this structure functions as a 6-port circulator, with symmetry group C_6. A useful property of this circuit is that it degenerates into two disconnected circulators when symmetry is destroyed by reversing some of the bias fields. Accordingly, it may be used as a five position waveguide switch [Fig. 11(b)], the output being selected by applying an appropriate set of bias fields to the six circulators. Similarly, an 8-port symmetrical ring circulator *with symmetry group* C_4 is obtained by connecting four 4-port star or turnstile circulators in a square array.

Discussion

The synthesis procedure described in this paper is particularly useful for the design of *symmetrical* circulators in the form of waveguide junctions containing ferrite obstacles.[18] A typical example of a circulator of

[17] Experimental results for circulators of this type are also reported by H. N. Chait and T. R. Curry in a paper, "A new Y-type circulator," to be published in *J. Appl. Phys.;* March, 1959.

[18] An unsymmetrical circulator of this type, based on an H-plane Tee structure, has been described by W. E. Swanson and G. J. Wheeler in a paper, "Tee Circulator," 1958 IRE WESCON Convention Record, vol. 2, pt. 1, pp. 151–156. This device has no symmetries, except for the trivial reflection symmetry in the H-plane, and a matching device is required in the side arm of the tee.

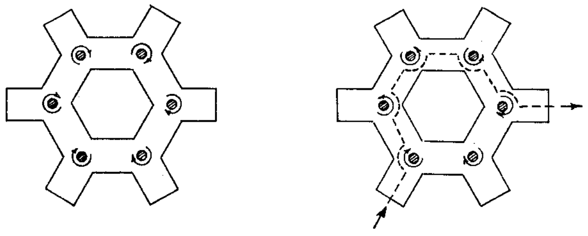


Fig. 11—(a) Six-port ring circulator. (b) Five-position waveguide switch.

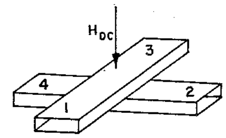


Fig. 12—Circulator with S_4 symmetry, using crossed guides coupled with a ferrite-loaded aperture.

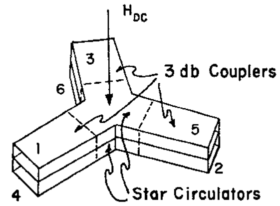


Fig. 13—Six-port circulator with C_{3h} (*class* (*b*)) symmetry, using two 3-port star circulators and three 3 db couplers.

this type is the 3-port star circulator shown in Fig. 5. Such circulators are compact and structurally simple, and have the important practical advantage of requiring only a small volume of ferrite material.

The specific applications which have been discussed in this paper have all used C_{nh} (*class* (*a*)) or C_n symmetry. Other circulators of novel design may be devised by using some of the alternative symmetries listed on Table I. For example, a circulator having S_4 symmetry may in principle be realized by coupling two crossed guides with a ferrite-loaded aperture, the bias field being applied along the axis of symmetry (Fig. 12). The coupling arrangement might conveniently take the form of a ferrite post passing through the coupling hole and extending across the full height of both waveguides. Another example, with C_{3h} (*class* (*b*)) symmetry, is shown in Fig. 13. A wave with amplitude a incident at port 1 passes through the 3 db coupler and divides into incident waves with amplitudes $0.707a$ and $j.0.707a$ in arms 1 and 4, respectively. These waves pass through the star circulators into arms 5 and 2 and combine in the 3 db coupler to give a transmitted wave of amplitude a at port 2. Similarly, a wave incident at port 2 emerges at port 3, etc. The scattering matrix, therefore, performs a cyclic substitution on the incident wave amplitudes, and the junction functions as a circulator. This device is more compact than the 6-port ring circulator [Fig. 12(a)]. It also has the advantage of requiring only two ferrite posts, which may both be biased by the same magnet.

Applications of the synthesis procedure are not necessarily restricted to junctions of rectangular waveguide. For example, a 3-port star circulator might be realized in the form of an H-plane Y junction of fin-line waveguide. Such a structure would be expected to have a larger bandwidth than the corresponding rectangular waveguide structure. Furthermore, the volume of ferrite material required would be reduced, owing to the localization of the microwave fields. The adjustment procedure would be identical with that outlined above for the rectangular waveguide version. In similar fashion, 3-port circulators might be realized in the form of either Stripline or coaxial line Y junctions.

Appendix

Let S be the scattering matrix of a lossless 3-port junction with C_3 or C_{3h} symmetry. S is diagonalized by a unitary matrix U having for its columns the normalized eigenvectors (14). Then,

$$S = U\begin{bmatrix} s_0 & 0 & 0 \\ 0 & s_1 & 0 \\ 0 & 0 & s_{-1} \end{bmatrix} U^{-1}$$

where s_0, s_1, s_{-1} are the eigenvalues of S. When the right-hand side of this equation is multiplied out, the scattering coefficients are obtained as functions of the eigenvalues.

$$S_{11} = S_{22} = S_{33} = \frac{s_0 + s_1 + s_{-1}}{3}$$

$$S_{12} = S_{23} = S_{31} = \frac{s_0 + \epsilon^{j2\pi/3}s_1 + \epsilon^{-j2\pi/3}s_{-1}}{3}$$

$$S_{13} = S_{32} = S_{21} = \frac{s_0 + \epsilon^{-j2\pi/3}s_1 + \epsilon^{j2\pi/3}s_{-1}}{3}.$$

Similarly, the scattering coefficients of a lossless 4-port junction with C_4, C_{4h}, or S_4 symmetry are given as functions of the eigenvalues s_0, s_1, s_{-1}, s_2 by the equations,

$$S_{11} = S_{22} = S_{33} = S_{44} = \frac{s_0 + s_1 + s_{-1} + s_2}{4}$$

$$S_{12} = S_{23} = S_{34} = S_{41} = \frac{s_0 + js_1 - js_{-1} - s_2}{4}$$

$$S_{13} = S_{31} = S_{24} = S_{42} = \frac{s_0 - s_1 - s_{-1} + s_2}{4}$$

$$S_{14} = S_{43} = S_{32} = S_{21} = \frac{s_0 - js_1 + js_{-1} - s_2}{4}.$$